1. a_1, a_2, a_3, a_4, a_5 is an arithmetic sequence of real numbers whose terms sum to 40. Exactly one of these terms is uniquely determined by this information. What is its value?

   Answer: 8

   Solution:

   This is an arithmetic sequence, so let k denote their common difference. Then

   $$\begin{aligned} a_1 &= a_3 - 2k \\ a_2 &= a_3 - k \\ a_3 &= a_3 \\ a_4 &= a_3 + k \\ a_5 &= a_3 + 2k. \end{aligned}$$

   Hence, the sum is $40 = (a_3 - 2k) + (a_3 - k) + (a_3) + (a_3 + k) + (a_3 + 2k) = 5a_3$, which implies that $a_3 = \boxed{8}$.

2. Given complex numbers $z_1 = 3 + 4i, z_2 = 5i, z_3 = 3 - 4i$, compute the positive real number z such that the expression $\left(\frac{z_1 - z_2}{z_1 - z}\right) \cdot \left(\frac{z_3 - z}{z_3 - z_2}\right)$ is real.

   Answer: 5

   Solution: Plugging in values, we get $\left(\frac{3 - i}{3 + 4i - z}\right) \cdot \left(\frac{3 - 4i - z}{3 - 9i}\right)$. Upon rationalizing the denominator, we get $\frac{(3 - i)(1 + 3i)((3 - z) - 4i)^2}{30(3 - z)^2 + 16)}$. Collecting terms with a factor of i in the numerator yields the equation $(3 - z)^2 + 6(3 - z) - 16 = 0 \Rightarrow (3 - z)^2 + 6(3 - z) + 9 = 25 \Rightarrow z^2 = 25 \Rightarrow z = \boxed{5}$.

3. Let $f(x) = ax^2 + bx + c$ where $a \neq 0$. Find d, where $0 < d < 1$, such that $f(0) = 2014$, $f(d^2) = 2015$, $f(d) = 2016$, and the sum of the roots of f is 0.

   Answer: $\frac{1}{\sqrt{2}}$

   Solution: The sum of the roots of a quadratic is $-\frac{b}{a}$, so the sum of the roots of f equals 0 if and only if $b = 0$. Now, the three values of f give us

   $$\begin{aligned} c &= 2014 \\ ad^4 + bd^2 + c &= 2015 \\ ad^2 + bd + c &= 2016 \end{aligned}$$

   Subtracting the first equation from the second two and substituting $b = 0$, we obtain

   $$\begin{aligned} ad^4 &= 1 \\ ad^2 &= 2 \end{aligned}$$

   Dividing equation one by equation two, we get that

   $$d^2 = \frac{1}{2}$$

   Therefore, it follows that $d = \pm\frac{1}{\sqrt{2}}$. Since we want $0 < d < 1$, we conclude that $\boxed{d = \frac{1}{\sqrt{2}}}$.

4. Suppose a sequence $\{a_n\}$ of real numbers follows the rule $a_n = p(n)$, wehere p is a polynomial with real coefficients of degree less than or equal to 6. If $\{a_1, a_2, \cdots, a_8\} =$ $\{-2, -93, -458, -899, 366, 8623, 35302, 101337\}$, what is a_9?

   Answer: 241246

   Solution: Since p is a polynomial, and we know its value at points that form an arithmetic sequence if we iterate the process of taking consecutive differences n times, the differences are constant. Once we have these differences, we can propogate the next value to get a_9.

5. Lynnelle and Moor are playing a game of Set. In Set, there are 27 red cards, 27 purple cards, and 27 green cards and at the end of the game, all the cards are divided between the two players. At the end of the game, the number of red cards Lynnelle has is the same as the number of green cards Moor has. We also know that Lynnelle has 17 more cards than Moor at the end of the game. How many purple cards does Lynnelle have?

   Answer: 22

   Solution: Let r, p, g be the number of red, purple, and green cards Moor has respectively. Then Lynnelle has $27 - r, 27 - p, 27 - g$ red, purple, and green cards. Since the number of red cards Lynnelle has is the same as the number of Moor's green cards, we have that $27 - r = g$. Thus, $r+g = 27$. Next, Lynnelle has 17 more cards than Moor so $(27-r)+(27-p)+(27-g)-(r+p+g) = 17$. Thus, $2(r + p + g) = 81 - 17 = 64$ so $r + p + g = 32$. Now, we know from earlier that $r + g = 27$ so this means that $p + 27 = 32$ and hence $p = 5$.

   Therefore, Lynnelle has $27 - p = \boxed{22}$ purple cards.

6. Compute
$$\sum_{m=1}^{2016} \sum_{k=m-2016}^{m-2} \frac{1}{m^2 + k^2 - 2mk - m + k}$$

   Answer: 2015

   Solution: Note that the denominator of the fraction can be factored. Then, if we make the substitution $n = m - k$, then the entire inner sum can be written without m, and becomes a simple telescoping sum. The answer follows.

$$\begin{aligned}
\sum_{m=1}^{2016} \sum_{k=m-2016}^{m-2} \frac{1}{m^2 + k^2 - 2mk - m + k} &= \sum_{m=1}^{2016} \sum_{k=m-2016}^{m-2} \frac{1}{(m-k)(m-k-1)} \\
&= \sum_{m=1}^{2016} \sum_{n=2}^{2016} \frac{1}{n(n-1)} \\
&= 2015 \sum_{n=2}^{2016} \frac{1}{n(n-1)} \\
&= 2015 \cdot \frac{2015}{2016} \\
&= \boxed{2015}
\end{aligned}$$

7. Find the real roots of $x^4 + 4x^3 + 6x^2 + 4x - 15$.

   Answer: $\mathbf{1, -3}$

Solution: Note that $x^4 + 4x^3 + 6x^2 + 4x - 15 = (x+1)^4 - 2^4 = ((x+1)^2 + 4)((x+1)^2 - 4)$. Thus, real the roots of $x^4 + 4x^3 + 6x^2 + 4x - 15$ are the real roots of $(x+1)^2 - 4$, so the roots are $x = \boxed{1, -3}$.

8. The polynomial $f(x) = x^3 - 4\sqrt{3}x^2 + 13x - 2\sqrt{3}$ has three real roots, a, b, and c. Find

$$\max\{a+b-c, a-b+c, -a+b+c\}.$$

Answer: $\mathbf{2\sqrt{3} + 2\sqrt{2}}$

Solution: The expressions we are trying to find the maximum of are very symmetric. This, in combination with knowledge of Viete's formulas motivates us to find a polynomial whose roots are $a+b-c, a-b+c, -a+b+c$. The following is what Viete gives us:

$$a+b+c = 4\sqrt{3}, \quad ab+bc+ca = 13, abc = 2\sqrt{3}.$$

We find that

$$(a+b-c) + (a-b+c) + (-a+b+c) = a+b+c = 4\sqrt{3}$$

and the cyclic sum

$$\sum_{cyc}(a+b-c)(a-b+c) = \sum_{cyc} a^2 - b^2 - c^2 + 2bc = -(a^2+b^2+c^2) + 2(ab+bc+ca) =$$

$$-(a+b+c)^2 + 4(ab+bc+ca) = -48 + 52 = 4.$$

This gives the x^2 and x coefficients of the polynomial we seek. Finally,

$$(a+b-c)(a-b+c)(-a+b+c) = (a^2 - b^2 - c^2 + 2bc)(-a+b+c) =$$

$$= -(a^3+b^3+c^3) + (a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2) - 2abc =$$

$$= -(a+b+c)^3 + 4(a^2b + \cdots + ca^2) + 4abc = -(a+b+c)^3 + 4(ab+bc+ca)(a+b+c) - 8abc =$$

$$-(4\sqrt{3})^3 + 4(4\sqrt{3})(13) - 8 \cdot 2\sqrt{3} = -192\sqrt{3} + 208\sqrt{3} - 16\sqrt{3} = 0.$$

Thus, $a+b-c, a-b+c, -a+b+c$ are the roots of

$$x^3 - 4\sqrt{3}x^3 + 4x = x(x^2 - 4\sqrt{3}x + 4$$

which has roots

$$x_1 = 0, x_{2,3} = \frac{4\sqrt{3} \pm \sqrt{(4\sqrt{3})^2 - 16}}{2} = \frac{4\sqrt{3} \pm \sqrt{32}}{2} = 2\sqrt{3} \pm 2\sqrt{2}.$$

Clearly, the largest of these three is $\boxed{2\sqrt{3} + 2\sqrt{2}.}$

9. Consider the path formed by an infinite number of line segments $L_1, L_2, ...$ of length $l_1, l_2, ...$, where $l_1 = 1$; L_1 starts at the origin and goes in the positive y direction; and for $i \geq 2$, L_i's start point is L_{i-1}'s end point, L_i is rotated 60° counterclockwise from L_{i-1}, and $l_i = k \cdot l_{i-1}$ for some constant $0 < k < 1$. The path looks like a hexagonal spiral that converges to a point. What are the coordinates for that point?

Answer: $\left(\frac{k+k^2}{2+2k^3}, \frac{2+k\sqrt{3}-k^2\sqrt{3}}{2+2k^3}\right)$

Solution: There are 6 types of segments: type 1: $(0, l)$, type 2: $(l/2, \sqrt{3}l/2)$, type 3: $(l/2, -\sqrt{3}l/2)$, and their negatives. Only segments 2 and 3 contribute for the x coordinate, so it is

$$x = \sum_{j=0}^{\infty} (-1)^j \frac{k^{3j+1}}{2} + \sum_{j=0}^{\infty} (-1)^j \frac{k^{3j+2}}{2} = \left(\frac{k}{2} + \frac{k^2}{2}\right) \sum_{j=0}^{\infty} (-k^3)^j$$

which is a geometric series. $\boxed{x = \dfrac{k + k^2}{2 + 2k^3}}$.

Similarly for y,

$$y = \sum_{j=0}^{\infty} (-1)^j k^{3j} + \sum_{j=0}^{\infty} (-1)^j \frac{\sqrt{3}k^{3j+1}}{2} + \sum_{j=0}^{\infty} (-1)^j \frac{-\sqrt{3}k^{3j+2}}{2} = \left(1 + \frac{\sqrt{3}k}{2} + \frac{\sqrt{3}k^2}{2}\right) \sum_{j=0}^{\infty} (-k^3)^j$$

$$\boxed{y = \frac{2 + k\sqrt{3} - k^2\sqrt{3}}{2 + 2k^3}}$$

10. Let $X_1, X_2, X_3, ..$ be a sequence of strings of 0s and 1s derived in the following manner: $X_1 =$ "1", and X_{n+1} is formed by replacing every "0" in X_n with a "1", and replacing every "1" in X_n with "11000". Thus $X_1 =$ "1", $X_2 =$ "11000", $X_3 =$ "1100011000111", and so on. How many times does "01" occur in X_{2016}?

Answer: $\mathbf{\frac{3^{2015}-3}{4}}$

Solution: Let u_n be the number of "0"s in X_n, and let v_n be the number of "1"s in X_n. From the formula given for X_{n+1}, it is clear that the "0" from each occurence of "01" in X_{n+1} comes from a unique "1" in X_n, so there are at most v_n occurences of "01" in X_{n+1}. Furthermore, since "0" and "1" are both replaced by strings beginning with "1", for each "1" in X_n, if it is followed by either a "0" ("10") or a "1" ("11"), it produces a "01" instance in X_{n+1}. Thus the only way a "1" in X_n does not lead to a "01" in X_{n+1} is if it is the last digit in X_n, because there are no "1"s to follow the "11000". We conclude that "01" occurs v_n times in X_{n+1} if X_n ends with a "0", and it occurs $v_n - 1$ times when X_n ends with a "1". Since "1" is replaced by a string ending in "0" and "0" is replaced by a string ending in "1", it follows that X_n ends in a "1" for odd n and ends in a "0" for even n. Thus the occurences of "01" in X_n is v_{n-1} for n odd and $v_{n-1} - 1$ for n even.

To calculate v_n, we note that the formula for X_n tells us immediately that $u_{n+1} = 3v_n$ and $v_{n+1} = 2v_n + u_n = 2v_n + 3v_{n-1}$, and so

$$v_{n+1} = 2v_n + 3v_{n-1}.$$

Suppose a number a satisfies $a^2 = 2a + 3$. Then $a^3 = 2a^2 + 3a = 2(2a + 3) + 3a = 7a + 6$. By induction, it follows that $a^n = v_n a + D_n$. The value of D_n is irrelevant because there are 2 numbers a with this property, and they are the solutions to $a^2 - 2a - 3 = 0$, which are -1,3. Thus for each n, we have

$$3^n = 3v_n + D_n$$
$$(-1)^n = (-1)v_n + D_n,$$

where D_n is the same for both equations. Subtracting the second equation from the first, we get

$$3^n - (-1)^n = (3 - (-1))v_n + D_n - D_n,$$

which yields

$$v_n = \frac{3^n - (-1)^n}{3 - (-1)} = \frac{3^n - (-1)^n}{4}.$$

Since 2016 is even, we get that "01" occurs $v_{2015} - 1 = \frac{3^{2015}-3}{4}$ times in X_{2016}.