1. List all triples of positive integers $\{p_1, p_2, p_3\}$ where p_1, p_2, and p_3 are all prime, $p_1 < p_2 < p_3$, and p_1, p_2 differ by 2, p_2, p_3 differ by 2. For example $\{6, 8, 10\}$ is a triple that satisfies the last two properties, but not the first one, so it is not included in the answer.

   Answer: $\{3, 5, 7\}$

   Solution: Let a be the smallest element of such a triple, so the triple is $\{a, a+2, a+4\}$. Looking at the remainders mod 3, our triple becomes $\{a, a+2, a+1\}$, and it is clear that no matter what a is, exactly one element in the triple will be divisible by 3. Since the only prime divisible by 3 is 3, one of the numbers in the triple must be 3. $\{-1, 1, 3\}$ and $\{1, 3, 5\}$ do not work because 1 is not a prime (and -1 is not positive), but $\{3, 5, 7\}$ does, and it is the only triplet with the required properties, so $\{3, 5, 7\}$ is the answer.

2. An ant is walking on the edges of an icosahedron of side length 1. Compute the length of the longest path that he can take if he never crosses the same edge twice, but is allowed to revisit vertices.

   Answer: 25

   Solution: For any vertex except the starting and ending vertices, the ant can only visit 4 of the adjacent 5 edges. (The number of times the ant enters must be the number of times he exist, hence even.)

   There are 12 vertices, so the sum over all vertices of the number of adjacent edges the ant transverses is at most: $2 \cdot 5 + 4 \cdot 10 = \boxed{50}$.

   This counts each edge twice, so any such path has length bounded by 25. It remains to construct such a path.

3. Compute the number of trailing zeros of 2016!.

   Answer: 502

   Solution: Each trailing zero is formed from a factor of 2 and a factor of 5. There are more factors of 2 than factors of 5 in 2016!, so we need only to count the number of factors of 5.

   - $\frac{2016}{5} = 403.2$, so 403 numbers contribute one factor of 5
   - $\frac{2016}{25} = 80.6$, so of the 403 numbers that contribute one 5, 80 contribute another factor of 5.
   - $\frac{2016}{125} = 16.13$, so of the 80 numbers contributing 2 factors of 5, 16 contribute another one.
   - $\frac{2016}{625} = 3.226$, so of the 16 numbers contributing 3 factors of 5, 3 contribute another one.
   - $\frac{2016}{3125} < 1$, so no number less than or equal to 2016 contributes 5 factors of 5.

   We therefore have $403 + 80 + 16 + 3 = \boxed{502}$ trailing zeros.

4. A positive integer $n > 1$ is called *multiplicatively perfect* if the product of its proper divisors (divisors excluding the number itself) is n. For example, 6 is multiplicatively perfect since $6 = 1 \times 2 \times 3$. Compute the number of multiplicatively perfect integers less than 100.

   Answer: 32

   Solution: Let $n > 1$ be a multiplicatively perfect integer. Then we can write it in the form $n = mp$ where m is any integer (possibly 1) and p is prime. If $m = 1$, then n is prime and its only proper divisor is 1 so it cannot be multiplicatively perfect.

   Next, suppose $m = p$. Then $n = p^2$ for some prime p and its proper divisors are 1 and p. However, the product of 1 and p cannot equal p^2, so it follows that $m \neq p$. Since $m \neq 1$ and

$m \neq p$, the proper divisors of n must contain at least 1, m, and p. Since $1 \times m \times p = n$ already, n cannot have any other proper divisors, or else the product would be greater than n.

If r is a proper divisor of m, then it is also a proper divisor of n. Since we reasoned that the only proper divisors of n are $1, m, p$, it follows that either $r = 1$ or $r = p$ ($r \neq m$ since r is a proper divisor of m). Therefore, the only proper divisors of m are 1 or p so there are only two possibilities. Either $m = p^2$ or $m = q$ for some prime $q \neq p$. Thus, n is multiplicatively perfect if and only if $n = p^3$ or $n = pq$ for distinct primes p, q.

Counting all integers $1 < n < 100$ of the form p^3 or pq gives us $\boxed{32}$ multiplicatively perfect numbers less than 100.

5. Let $d(n)$ be the number of positive integer divisors of a positive integer n. For example, $d(6) = 4$, because the divisors of 6 are 1, 2, 3, and 6. Compute

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2},$$

given that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Answer: $\boldsymbol{\frac{\pi^4}{36}}$

Solution: We have that $d(n) = \sum\limits_{d|n} 1$, so $\sum\limits_{n=1}^{\infty} \frac{d(n)}{n^2} = \sum\limits_{n=1}^{\infty} \sum\limits_{d|n} \frac{1}{n^2} = \sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{\infty} \frac{1}{(ij)^2} = \sum\limits_{i=1}^{\infty} \frac{1}{i^2} \sum\limits_{j=1}^{\infty} \frac{1}{j^2} = \left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{36}$.

6. Suppose $n > 0$ is an integer which, when written in base 10, has all digits either 0 or 1. If 17 evenly divides n, find the smallest possible value of n.

Answer: 11101

Solution: We find this n in an algorithmic way. We start with the number 1 and construct a tree of remainders modulo 17; the first time we encounter a 0 we will get the smallest number which is divisible by 17 and which has decimal representation using only $0, 1$. This is the construction of the tree: each node containing the integer k splits off into two nodes, the left node contains $10k$ mod 17 and the right node contains $10k + 1$ mod 17. Whenever a node appears with a remainder that has already been seen, we do not have to continue computation for that node because it will only result in a larger number divisible by 17. If we call 1 the base of the tree, we find 0 at the location 2 right, 1 left, 1 right from the base. This corresponds to the number $\boxed{11101}$ because we started with 1 and as we traverse the tree, we append digits to the right of our number. Every time we move to the right we append a 1 and every time we move to the left we append a 0. Then number 11101 is indeed divisible by 17, and no smaller number is divisible by 17 because the numbers in the tree increase with each level and within levels they increase left to right.

7. Lennart and Eddy are playing a betting game. Lennart starts with 7 dollars and Eddy starts with 3 dollars. Each round, both Lennart and Eddy bet an amount equal to the amount of the player with the least money. For example, on the first round, both players will bet 3 dollars. A fair coin is then tossed. If it lands heads, Lennart wins all the money bet; if it lands tails, Eddy wins all the money bet. They continue playing this game until someone has no money. What is the probability that Eddy ends with 10 dollars?

Answer: $\mathbf{\frac{3}{10}}$.

Solution: We note that each player has expected winnings of 0 dollars per round. Therefore, each player has expected winnings of 0 dollars for the entire game. Note that Eddy can only end the game with 0 or 10 dollars, which are respectively -3 and 7 dollars in winnings. Since Eddy should have expected winnings of 0, his probability of winning 7 dollars is $\frac{3}{10}$.

Solution: There is a brute force solution. Let p_x denote the probability Eddy wins given that he has x dollars. Then, note that

$$\begin{aligned}
p_3 &= \frac{1}{2}p_6, \\
p_6 &= \frac{1}{2} + \frac{1}{2}p_2, \\
p_2 &= \frac{1}{2}p_4, \\
p_4 &= \frac{1}{2}p_8, \\
p_8 &= \frac{1}{2} + \frac{1}{2}p_6.
\end{aligned}$$

Doing lots of substitutions, we get that

$$p_6 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}p_6\right)\right)\right) = \frac{9}{16} + \frac{1}{16}p_6.$$

Therefore, $p_6 = \frac{3}{5}$ and $p_3 = \frac{3}{10}$.

8. Consider a 2011×2012 grid of points from $(1,1)$ to $(2011,2012)$ with the point $(1066,1453)$ removed. Starting at $(1,1)$ and only moving up or to the right at each step, compute the number of different ways you can get to $(2011,2012)$. You may express your answer using multiple binomial coefficients.

   Answer: $\boldsymbol{\binom{4021}{2010} - \binom{2517}{1065}\binom{1504}{945}}$

   Note that $\boldsymbol{\binom{4021}{2010} = \binom{4021}{2011}, \binom{2517}{1065} = \binom{2517}{1452}, \binom{1504}{945} = \binom{1504}{559}}$.

   Solution: Thinking of a path from $(1,1)$ to (m,n) as the $m+n-2$-term sequence where $m-1$ elements are Right and $n-1$ elements are Up, each possible sequence generates a different path, and every path can be represented by one such sequence. Thus, there are $\binom{m+n-2}{m-1}$ such paths. Letting $m = 2011$ and $n = 2012$ yields us $\binom{4021}{2010}$ from $(1,1)$ to $(2011,2012)$. From these paths we must subtract the ones that go through $(1066,1453)$. Any such first goes from $(1,1)$ to $(1066,1453)$ (with $\binom{2517}{1065}$ possibilities) and then must go from $(1066,1453)$ to $(2011,2012)$ (with $\binom{1504}{945}$ possibilities). Thus the answer is $\boxed{\binom{4021}{2010} - \binom{2517}{1065}\binom{1504}{559}}$.

9. Let $X_1, X_2, X_3, ..$ be a sequence of strings of 0s and 1s derived in the following manner: $X_1 =$ "1", and X_{n+1} is formed by replacing every "0" in X_n with a "1", and replacing every "1" in X_n with "11000". Thus $X_1 =$ "1", $X_2 =$ "11000", $X_3 =$ "1100011000111", and so on. How many times does "01" occur in X_{2016}?

   Answer: $\mathbf{\frac{3^{2015}-3}{4}}$

   Solution: Let u_n be the number of "0"s in X_n, and let v_n be the number of "1"s in X_n. From the formula given for X_{n+1}, it is clear that the "0" from each occurence of "01" in X_{n+1} comes from a unique "1" in X_n, so there are at most v_n occurences of "01" in X_{n+1}. Furthermore, since "0" and "1" are both replaced by strings beginning with "1", for each "1" in X_n, if it is followed by either a "0" ("10") or a "1" ("11"), it produces a "01" instance in X_{n+1}. Thus the only way a "1" in X_n does not lead to a "01" in X_{n+1} is if it is the last digit in X_n, because there are no "1"s to follow the "11000". We conclude that "01" occurs v_n times in X_{n+1} if X_n ends with a "0", and it occurs $v_n - 1$ times when X_n ends with a "1". Since "1" is replaced by a string ending in "0" and "0" is replaced by a string ending in "1", it follows that X_n ends in a "1" for odd n and ends in a "0" for even n. Thus the occurences of "01" in X_n is v_{n-1} for n odd and $v_{n-1} - 1$ for n even.

To calculate v_n, we note that the formula for X_n tells us immediately that $u_{n+1} = 3v_n$ and $v_{n+1} = 2v_n + u_n = 2v_n + 3v_{n-1}$, and so

$$v_{n+1} = 2v_n + 3v_{n-1}.$$

Suppose a number a satisfies $a^2 = 2a + 3$. Then $a^3 = 2a^2 + 3a = 2(2a + 3) + 3a = 7a + 6$. By induction, it follows that $a^n = v_n a + D_n$. The value of D_n is irrelevant because there are 2 numbers a with this property, and they are the solutions to $a^2 - 2a - 3 = 0$, which are -1,3. Thus for each n, we have

$$3^n = 3v_n + D_n$$
$$(-1)^n = (-1)v_n + D_n,$$

where D_n is the same for both equations. Subtracting the second equation from the first, we get

$$3^n - (-1)^n = (3 - (-1))v_n + D_n - D_n,$$

which yields

$$v_n = \frac{3^n - (-1)^n}{3 - (-1)} = \frac{3^n - (-1)^n}{4}.$$

Since 2016 is even, we get that "01" occurs $v_{2015} - 1 = \frac{3^{2015}-3}{4}$ times in X_{2016}.

10. A continuous real-valued function f on the positive real numbers has the property that for all positive x and y, $f(xy) = xf(y) + yf(x)$. Determine all such functions f.

Answer: $\boldsymbol{cx \log x}$

Solution: For any positive a and any positive integer n, $f(a^n) = na^{n-1}f(a)$, which may be verified by induction. The $n = 1$ case is trivially true. If $f(a^k) = ka^{k-1}f(a)$ for some positive integer k, then $f(a^{k+1}) = f(a^k \cdot a) = a^k f(a) + af(a^k) = (k + 1)a^k f(a)$. $f(1) = f(1 \cdot 1) = 1f(1) + 1f(1) = 2f(1)$, so $f(1) = 0$. For any positive a, $0 = f(1) = f(a \cdot \frac{1}{a}) = af(\frac{1}{a}) + \frac{1}{a}f(a)$, so $f(\frac{1}{a}) = -\frac{1}{a^2}f(a)$. Thus $f(a^{-n}) = f((\frac{1}{a})^n) = n(\frac{1}{a})^{n-1}f(\frac{1}{a}) = na^{-n+1}(-\frac{1}{a^2}f(a)) = -na^{-n-1}f(a)$. Therefore, for all positive a and all integers n, $f(a^n) = na^{n-1}f(a)$. Additionally, $f(a) = f((a^{\frac{1}{n}})^n) = n(a^{\frac{1}{n}})^{n-1}f(a^{\frac{1}{n}}) = na^{1-\frac{1}{n}}f(a^{\frac{1}{n}})$, so $f(a^{\frac{1}{n}}) = \frac{1}{n}a^{\frac{1}{n}-1}f(a)$. Thus for all rational numbers $q = \frac{m}{n}$, where m and n are integers, and all positive a, $f(a^q) = f(a^{\frac{m}{n}}) = f((a^{\frac{1}{n}})^m) = m(a^{\frac{1}{n}})^{m-1}f(a^{\frac{1}{n}}) = ma^{\frac{m}{n}-\frac{1}{n}}(\frac{1}{n}a^{\frac{1}{n}-1}f(a)) = \frac{m}{n}a^{\frac{m}{n}-1}f(a) = qa^{q-1}f(a)$. Because f is continuous, a^r is continuous in r for any fixed positive a, and any real number can be approximated to arbitrary precision by rational numbers (i.e. any real number can be written as the limit of a sequence of rational numbers), for all real numbers r and all positive real numbers a, $f(a^r) = ra^{r-1}f(a)$. Therefore, for all positive x, $f(x) = f(e^{\log x}) = (\log x)e^{\log x-1}f(e) = x\log x \cdot \frac{f(e)}{e} = cx\log x$, where $c = \frac{f(e)}{e}$ can be any real constant.