1. Two externally tangent unit circles are constructed inside square $ABCD$, one tangent to AB and AD, the other to BC and CD. Compute the length of AB.

   Answer: $\mathbf{2+\sqrt{2}}$

   Solution: Observe that the diagonal of the square has length $2+2\sqrt{2}$. Therefore, the square has side length $\boxed{2+\sqrt{2}}$

2. We say that a triple of integers (a, b, c) is *sorted* if $a < b < c$. How many sorted triples of positive integers are there such that $c \leq 15$ and the greatest common divisor of a, b, and c is greater than 1?

   Answer: 46

   Solution: If the greatest common divisor is not 1, then it must be 2 or 3 or 4 or 5. If it is 2 or 4, there are $\binom{7}{3} = 35$ such triples. If it is 3, there is $\binom{5}{3} = 10$ such triples. If it is 5, there is $\binom{3}{3} = 1$ such triple. Therefore, there are a total of $35 + 10 + 1 = \boxed{46}$ such triples.

3. Two players play a game where they alternate taking a positive integer N and decreasing it by some divisor n of N such that $n < N$. For example, if one player is given $N = 15$, they can choose $n = 3$ and give the other player $N - n = 15 - 3 = 12$. A player loses if they are given $N = 1$.

   For how many of the first 2015 positive integers is the player who moves first guaranteed to win, given optimal play from both players?

   Answer: 1007.

   Solution:

   Define a losing position for Player 1 as a position where all moves will allow Player 2 to win. Define a winning position for Player 1 as a position where there exists a move which puts Player 2 into a losing position. After testing some small cases, we hypothesize the winning positions for Player 1 are exactly those where N is even. We can prove this using strong induction.

   The base case is simple. By definition, Player 1 loses when $N = 1$.

   For the inductive step, assume that all positions $P < N$, P is a losing position if P is odd and is a winning position if P is even. We must show the same holds for N. If N is even, then Player 1 can decrease N by 1, giving Player 2 the number $N - 1$, which is a losing position by our inductive hypothesis. Therefore, if N is even, N is a winning position. If N is odd, then all factors of N are odd, so any move Player 1 makes will give Player 2 an even number, which, by our inductive hypothesis, is a winning position. Therefore, if N is odd, N is a losing position. This completes the inductive step.

   Because all even numbers are winning positions for Player 1, there are $\lfloor \frac{2015}{2} \rfloor = \boxed{1007}$ winning positions.

4. The polynomial $x^3 - 2015x^2 + mx + n$ has integer coefficients and has 3 distinct positive integer roots. One of the roots is the product of the two other roots. How many possible values are there for n?

   Answer: 16

   Solution: Let the roots be r, s, and rs. By Vieta's formulas, we see that $r + s + rs = 2015$ and

$n = -r^2s^2$. Note that since r and s are positive integers, we get:

$$\begin{aligned} r + s + rs &= 2015 \\ 1 + r + s + rs &= 2016 \\ (1 + r)(1 + s) &= 2016. \end{aligned}$$

We factor 2016 to get $2^5 \cdot 3^2 \cdot 7$. We see that since 2016 has $6 \cdot 3 \cdot 2$ distinct factors, there are 36 possible pairs for r and s. However, we count each of these twice, so there are $36/2 = 18$ distinct pairs for r and s. We need to exclude the case $r = 0$ because we only want positive roots. Also, if $r = 1$, we no longer have distinct roots because $rs = s$. Therefore, there are $18 - 2 = \boxed{16}$ possible values that $n = -r^2s^2$ can take.

5. You have a robot. Each morning the robot performs one of four actions, each with probability 1/4:

   - Nothing.
   - Self-destruct.
   - Create one clone.
   - Create two clones.

   Compute the probability that you eventually have no robots.

   Answer: $\mathbf{\sqrt{2} - 1}$

   Solution: Let $p(n)$ designate the probability that, starting with a group of n robots, eventually all robots in this group die off. One first sees that $p(n) = p(1)^n$.

   With this observation, we can just write down $p(1) = 1/4 + 1/4p(1) + 1/4p(2) + 1/4p(3) = 1/4(1 + p(1) + p(1)^2 + p(1)^3)$. Setting $p = p(1)$, we obtain the equation

   $$p = \frac{1}{4}\left(1 + p + p^2 + p^3\right).$$

   $p = 1$ is a solution but clearly does not satisfy the problem. There is a negative root, but probabilities aren't negative. So the answer is the positive root < 1, which is $\boxed{\sqrt{2} - 1}$.

6. Four spheres of radius 1 are mutually tangent. What is the radius of the smallest sphere containing them?

   Answer: $\mathbf{1 + \sqrt{\frac{3}{2}}}$

   Solution: Call the containing sphere S and the smaller spheres s_i for $i = 1, 2, 3, 4$. By symmetry of the arrangement S is tangent to each s_i, and the point of tangency must be the point on the sphere s_i farthest away from the center of mass of all 4 small spheres s_i. Said another way, if three of the spheres are placed on a table with the fourth on top of them then the point of tangency on the top sphere is the heighest point from the table. So it sufices to find the distance of this point from the center of mass. Because the spheres are tangent their centers define a tetrahedron of side length 2 by symmetry. The center of mass (CM) of the spheres is also the center of mass of the tetrahedron. We seek the distance from CM to a vertex (a sphere center). Because it is a center of mass, the sum of vectors from CM to each vertex is 0. Therefore one vector's length equals 3 times the projection of the other vectors onto it. Thus, if $\phi > \pi/2$ is the angle between two of the vectors and l is the vector length,

   $$l = 3l\cos(\pi - \phi) \implies \cos(\phi) = -1/3$$

Thus by the law of cosines and the side length of the tetrahedron the vector length is:

$$2^2 = 2l^2 - 2l^2\cos(\phi) \implies l = \sqrt{\frac{3}{2}}$$

So the radius of S is $1 + \sqrt{\frac{3}{2}}$.

7. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

Answer: $\frac{2\sqrt{2}-1}{2}$

Solution: First, note that the center of such a circle must lie on the y-axis and the circle must be tangent to the x-axis and the parabola. Now, consider a circle with center $(0, c)$ and radius r. Since the circle is tangent to the x-axis, necessarily $c = r$. The top half of the circle thus has equation $y = r + \sqrt{r^2 - x^2}$. For the circle to be tangent to the parabola, there needs to be exactly 2 solutions to the equation $r + \sqrt{r^2 - x^2} = 2 - x^2$. We compute:

$$\begin{aligned} r + \sqrt{r^2 - x^2} &= 2 - x^2 \\ r^2 - x^2 &= (2 - r - x^2)^2 \\ r^2 - x^2 &= x^4 - 2(2-r)x^2 + (2-r)^2 \\ x^4 + (2r-3)x^2 + (4-4r) &= 0 \end{aligned}$$

This gives a quadratic in x^2 so the determinant must be zero. Thus, $(2r-3)^2 + 16r - 16 = 0$. This simplifies to $4r^2 + 4r - 7 = 0$ and hence $r = \frac{-4 \pm \sqrt{128}}{8} = \frac{-1 \pm 2\sqrt{2}}{2}$. Since $r > 0$, it follows that the maximum radius is $\boxed{\frac{2\sqrt{2}-1}{2}}$.

8. For some nonzero constant a, let $f(x) = e^{ax}$ and $g(x) = \frac{1}{a}\log x$. Find all possible values of a such that the graphs of f and g are tangent at exactly one point.

Answer: $\left\{\frac{1}{e}, -e\right\}$.

Note that f and g are inverses of each other, so any intersection point must lie on the line $y = x$. We can prove this by noticing that $f(x) = g(x) \implies f(f(x)) = x$. If f is monotonically increasing (i.e. $a > 0$), $f(x) > x \implies f(f(x)) > f(x) > x$, and $f(x) < x \implies f(f(x)) < f(x) < x$, so we must have $f(x) = x$. The same argument with the inequalities reversed works if f is monotonically decreasing (i.e. $a < 0$).

Moreover, since f and g are tangent at this point, their derivatives must be equal. But since they are inverses, their derivatives are also reciprocals, so the slope m of the tangent line satisfies $m = \frac{1}{m} \iff m = \pm 1$. We try both cases.

First, by looking at f, $m = 1$ yields the system

$$\begin{aligned} e^{ax} &= x \\ ae^{ax} &= 1. \end{aligned}$$

Solving this gives $x = e$, $a = \frac{1}{e}$.

Second, we consider $m = -1$, which yields

$$\begin{aligned} e^{ax} &= x \\ ae^{ax} &= -1. \end{aligned}$$

Solving this gives $x = \frac{1}{e}, a = -e$.

Hence, report $\boxed{\left\{\frac{1}{e}, -e\right\}}$.

9. Consider a regular pentagon and connect each vertex to the pair of vertices farthest from it by line segments. The line segments intersect at 5 points to form another smaller pentagon. If the large pentagon has side length 1, compute the area of the smaller pentagon. Express your answer without trigonometric functions.

Answer: $\mathbf{\frac{1}{8}\sqrt{250 - 110\sqrt{5}}}$

Solution: It is clear by symmetry the the smaller pentagon is also regular. Let the original pentagon have vertices $ABCDE$. Label the smaller pentagon with vertices $A'B'C'D'E'$ where the primed vertex, X', is the single primed vertex farthest away from the unprimed one, X. Call the lengths $A'B' = x$ and $AC' = y$. It is clear from similar triangles that $\frac{AB}{A'B'} = \frac{AC}{B'C}$ or $\frac{1}{x} = \frac{2y+x}{y}$. But it is also clear that $x + y = 1$. This can be seen by the congruent triangles, $\triangle ABC$ and $\triangle AB'C$, because $AD \parallel BC$. Thus, $x = \frac{1}{2}\left(3 - \sqrt{5}\right), y = \frac{1}{2}\left(\sqrt{5} - 1\right)$.

It is easy to see that the area of a regular pentagon of sidelength d can be calculated by summing the areas of triangles formed with the center and two vertices. The height (apothem) is $d/2 \cot(2\pi/10)$. So the area of the whole pentagon is $5 \times \frac{1}{2} \times d/2 \cot(2\pi/10) \times d$. So the small pentagon area is:

$$\frac{5}{4}\cot(\pi/5)\left(\frac{1}{2}\left(3 - \sqrt{5}\right)\right)^2 = \frac{5}{8}\left(7 - 3\sqrt{5}\right)\cot(\pi/5)$$

And it remains to calculate $\cot(\pi/5)$. Notice that $\pi/5$ radians is actually 36°. So we can instead compute $\cot(36^\circ)$. For this is suffices to calculate $\cos(36^\circ)$. Using addition formulas,

$$\begin{aligned}1 = \cos(90^\circ) &= \cos(72^\circ + 18^\circ) = \cos 72^\circ \cos 18^\circ - \sin 72^\circ \sin 18^\circ \\ &= (2\cos^2 36^\circ - 1)\sqrt{\tfrac{1+\cos 36^\circ}{2}} - 2\sin 36^\circ \cos 36^\circ \sqrt{\tfrac{1-\cos 36^\circ}{2}}\end{aligned}$$

Then $\cos(36^\circ) = u$ satisfies the equation:

$$0 = (2u^2 - 1)\sqrt{\tfrac{1+u}{2}} - 2\sqrt{1-u^2} \cdot u\sqrt{\tfrac{1-u}{2}}$$

Because $0 < u < 1$ this can be simplified,

$$\begin{aligned}2\sqrt{1+u}\sqrt{1-u} \cdot u\sqrt{1-u} &= (2u^2 - 1)\sqrt{1+u} \\ 2u(1-u) &= 2u^2 - 1\end{aligned}$$

So $u = \frac{1+\sqrt{5}}{4}$. Thus the cotangent is $\frac{\frac{1+\sqrt{5}}{4}}{\sqrt{1-\left(\frac{1+\sqrt{5}}{4}\right)^2}} = \sqrt{1 + \frac{2}{\sqrt{5}}}$. So the area becomes,

$$\frac{5}{8}\left(7 - 3\sqrt{5}\right)\sqrt{1 + \frac{2}{\sqrt{5}}} = \frac{1}{8}\sqrt{250 - 110\sqrt{5}}$$

10. Let $f(x)$ be a function that satisfies $f(x)f(2-x) = x^2 f(x-2)$ and $f(1) = \frac{1}{403}$. Compute $f(2015)$.

Answer: 25

Solution: Plugging in $x = 2k+1$, we obtain

$$f(2k+1)f(-(2k-1)) = (2k+1)^2 f(2k-1)$$
$$f(2k+1) = (2k+1)^2 \frac{f(2k-1)}{f(-(2k-1))}$$

and plugging in $x = -(2k-1)$, we obtain

$$f(-(2k-1))f(2k+1) = (2k-1)^2 f(-(2k+1))$$
$$f(-(2k+1)) = \frac{(2k+1)^2}{(2k-1)^2} f(2k-1)$$

Now, note that obviously $f(1) = f(1)$ and plugging in $x = 1$ gives $f(-1) = (f(1))^2$. Inductively using the above formulas to compute $f(3), f(-3), f(5), f(-5), \ldots$, we see that $f(2k+1)$ and $f(-(2k+1))$ equal $(2k+1)^2$ times a rational function of $f(1)$. It turns out that the rational function of $f(1)$ part is periodic, with period 6:

$$f(3) = 3^2 \cdot \frac{1}{f(1)}$$
$$f(-3) = 3^2 \cdot f(1)$$
$$f(5) = 5^2 \cdot \frac{1}{(f(1))^2}$$
$$f(-5) = 5^2 \cdot \frac{1}{f(1)}$$
$$f(7) = 7^2 \cdot \frac{1}{f(1)}$$
$$f(-7) = 7^2 \cdot \frac{1}{(f(1))^2}$$
$$f(9) = 9^2 \cdot f(1)$$
$$f(-9) = 9^2 \cdot \frac{1}{f(1)}$$
$$f(11) = 11^2 \cdot (f(1))^2$$
$$f(-11) = 11^2 \cdot f(1)$$
$$f(13) = 13^2 \cdot f(1)$$
$$f(-13) = 13^2 \cdot (f(1))^2$$

Thus, we see that $\frac{f(2k+1)}{(2k+1)^2} = \frac{f(2(k+6)+1)}{(2(k+6)+1)^2}$ it is periodic in k with period 6. Since $2015 = 2 \cdot 1007 + 1$ and $1007 \equiv 5 \pmod 6$. It follows that $\frac{f(2015)}{2015^2} = \frac{f(2\cdot5+1)}{(2\cdot5+1)^2} = \frac{f(11)}{11^2}$.

As computed above, $f(11) = 11^2 \cdot (f(1))^2$ so $f(2015) = (2015 f(1))^2 = \left(\frac{2015}{403}\right)^2 = \boxed{25}$.

11. You are playing a game on the number line. At the beginning of the game, every real number on $[0, 4)$ is uncovered, and the rest are covered. A turn consists of picking a real number r such

that, for all x where $r \le x < r+1$, x is uncovered. The turn ends by covering all such x. At the beginning of a turn, one selects such a real r uniformly at random from among all possible choices for r; the game ends when no such r exists. Compute the expected number of turns that will take place during this game.

Answer: $\mathbf{\dfrac{11-4\ln 2}{3}}$

Solution: Let $f(n)$ be the expected number of turns that can be played on an interval of length n. When $n < 2$, we have that $f(n) = \lfloor n \rfloor$, obviously. When $n \ge 2$, consider the first move. r will be chosen uniformly from an interval of length $n-1$, so that we will be left with two regions to play in, one with length r, the other with length $n-r-1$. The number of moves we can expect to make then, given r, is $1 + f(r) + f((n-1)-r)$. Averaging over r, we have $f(n) = 1 + \frac{1}{n-1}\int_0^{n-1} f(r)\,dr + \frac{1}{n-1}\int_0^{n-1} f((n-1)-r)\,dr = 1 + \frac{2}{n-1}\int_0^{n-1} f(r)\,dr$. We can therefore compute $f(n)$ explicitly when $2 \le n \le 3$, and can therefore also compute $f(n)$ explicitly when $3 \le n \le 4$, to get $f(4) = \boxed{\dfrac{11-4\ln 2}{3}}$.

12. Consider the recurrence:

$$a_{n+1} = 4a_n(1-a_n)$$

Call a point $a_0 \in [0,1]$ $q-$periodic if $a_q = a_0$. For example, $a_0 = 0$ is always a $q-$periodic fixed point for any q. Compute the number of positive 2015$-$periodic fixed points.

Answer: $\mathbf{2^{2015}-1}$

Solution: The recurrence can be solved exactly. Make the substitution, $a_n = \sin^2 \pi\theta_n$, which works because we know that $a_n \in [0,1]$ for all n (To see this, note that the RHS is a parabola with maximum of height 1).

$\sin^2 \pi\theta_{n+1} = 4\sin^2 \pi\theta_n(1-\sin^2\pi\theta_n) = (2\sin\pi\theta_n\cos\pi\theta_n)^2 \implies \sin\pi\theta_{n+1} = \sin 2\pi\theta_n \implies \theta_n = \theta_0 2^n$.

In particular, $a_q = \sin^2\left(\arcsin(\sqrt{a_0})2^q\right)$ solves the original recurrence. But this is actually a degree-2^q polynomial in a_0, call it $f(x) = \sin^2\left(\arcsin(\sqrt{x})2^q\right)$. We can see this by composing the RHS of the original recurrence with itself q times. All of this function's 2^{q-1} maxima are of height 1 and are in [0,1]. It also has $2^{q-1}+1$ zeros in [0,1] two of which are on the ends on the interval. So it intersects $y = x$ at least 2^q fixed points and at most $2^q + 1$. But f is a degree-2^q polynomial and so is $x - f(x)$ so the latter case is not possible because such a polynomial has at most 2^q roots. Plugging in $q = 2015$ and removing 0 gives us $\boxed{2^{2015}-1}$ 2015-fixed points.

One does not actually need to solve the recurrence to do this problem. It suffices to notice that the function f is a polynomial, as described above, with 2^{q-1} maxima are of height 1. You can see this by looking at the RHS, $4a_n(1-a_n)$. This is a parabola of height 1 so it is continuous and onto [0,1]. Additionally 1/2 is mapped to a maximum and a maximum is mapped to 0. So after q iterations the desired result follows, the function acquires another bend each time.

13. Let $a, b, c \in \{-1, 1\}$. Evaluate the following expression, where the sum is taken over all possible choices of a, b, and c:

$$\sum abc(2^{\frac{1}{5}} + a2^{\frac{2}{5}} + b2^{\frac{3}{5}} + c2^{\frac{4}{5}})^4.$$

Answer: **768**

Solution: Replace $2^{i/5}$ with x_{i-1} and let the sum be $f(x_0, x_1, x_2, x_3)$. Notice that f is symmetric in its variables; swapping x_i wth x_j does not change the result of the summation. Next, we compute

$$f(0, x_1, x_2, x_3) = \sum abc(ax_1 + bx_2 + cx_3)^4.$$

But note that a particular choice of (a, b, c) gives the same fourth power as $(-a, -b, -c)$ but their products have different signs. In other words, we can split the sum into pairs of the form

$$(ax_1 + bx_2 + cx_3)^4 - (-ax_1 - bx_2 - cx_3)^4 = 0$$

and we conclude that $f(0, x_1, x_2, x_3) = 0$. Because f is symmetric, we conclude that $f = 0$ for $x_i = 0$ for any i. Then, because f is a polynomial in each x_i, x_i must divide f for every i. But the product $x_0x_1x_2x_3$ has overall degree 4, and because f is the sum of fourth powers, it must have overall degree at most 4. We conclude that

$$f(x_0, x_1, x_2, x_3) = Kx_0x_1x_2x_3$$

where K is a constant. All we must do is find K. But observe that, when we expand $abc(x_0 + ax_1 + bx_2 + cx_3)^4$, the $x_0x_1x_2x_3$ term has coefficient $4!(abc)^2 = 24$. Because there are 8 such terms in the sum, we conclude that

$$f(x_0, x_1, x_2, x_3) = 8 \cdot 24x_0x_1x_2x_3 = 192x_0x_1x_2x_3.$$

All that remains is to plug in the given values of x_0, x_1, x_2, x_3 to get $192 \cdot 2^{10/5} = 192 \cdot 4 = \boxed{768}$.

14. A small circle A of radius $\frac{1}{3}$ rotates, without slipping, inside and tangent to a unit circle B. Let p be a fixed point on A, and compute the length of the closed curve traced out by p as A rotates inside B.

Answer: $\mathbf{\frac{16}{3}}$

Solution: The curve traced is a deltoid. First of all it is clear that the radius of the small circle is $\frac{1}{3}$. It is easy to write equations for the point's trajectory using polar coordinates. Let θ be the polar angle in the frame of A (from its center) and ϕ be similar for B. The center of A is always at a radius $\frac{2}{3}$ from the center of B. So the trajectory of the center of A is $2/3(\cos\phi, \sin\phi)$. The coordinate of the point on A moves in a circle of its radius in the opposite direction, $1/3(-\cos\theta, \sin\theta)$. Because the point starts at a double tangency there is no phase differnce between the two and from simple geometry $\theta = 2\phi$. Therefore the point's trajectory is:

$$x = \frac{2}{3}\cos(\phi) - \frac{1}{3}\cos(2\phi)$$
$$y = \frac{2}{3}\sin(\phi) + \frac{1}{3}\sin(2\phi)$$

To compute the arc length we could try to eliminate the parameter but this is highly tedious. The correct way is to calculate the length in polar coordinates. Applying the well known formula:

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2} d\phi = \int_0^{2\pi} \sqrt{\left(\frac{-2}{3}\sin(\phi) + \frac{2}{3}\sin(2\phi)\right)^2 + \left(\frac{2}{3}\cos(\phi) + \frac{2}{3}\cos(2\phi)\right)^2} d\phi$$

$$= \frac{2}{3}\int_0^{2\pi} 2\sqrt{\cos^2\left(\frac{3\phi}{2}\right)} d\phi = \frac{16}{3}$$

15. Let x_1, x_2, x_3, x_4, x_5 be distinct positive integers such that $x_1+x_2+x_3+x_4+x_5 = 100$. Compute the maximum value of the expression

$$\frac{(x_2x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_2-x_1)(x_3-x_1)(x_4-x_1)} + \frac{(x_1x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_1-x_2)(x_3-x_2)(x_4-x_2)} + \frac{(x_1x_5+1)(x_2x_5+1)(x_4x_5+1)}{(x_1-x_3)(x_2-x_3)(x_4-x_3)} + \frac{(x_1x_5+1)(x_2x_5+1)(x_3x_5+1)}{(x_1-x_4)(x_2-x_4)(x_3-x_4)}.$$

Answer: 729000

Solution: The given expression is cyclic in x_1, x_2, x_3, x_4 so clearly x_5 plays some special role. One definite goal is to determine a way of eliminating (or at the very least, not being bothered by) the denominators of this expression. Toying around with some ideas, we see that each denominator is a product of $(x_i - x_j)$ for fixed i and $j \neq i$ (and $j \neq 5$). So what if instead we looked at $p_i(x) = \prod_{k=1,k\neq i}^{4}(x_k - x)$? Then we have the denominators are $\frac{1}{p_i(x_i)}$. In a leap of faith, we look at the expression $\frac{p_i(x)}{p_i(x_i)}$. This is equal to 1 when we plug in x_i and is equal to 0 when we plug in x_j for $j \neq i, 5$. Then the polynomial $p(x) = \sum_{i=1}^{4} \frac{p_i(x)}{p_i(x_i)}$ has $p(x_i) = 1$ for $1 \leq i \leq 4$. Note that $p(x)$ has degree at most 3- hence it is identically equal to 1, because $p(x)-1$ has 4 roots. If we plug in $x = \frac{1}{x_5}$ and multiply both sides of the equation by x_5^3 we get that the given expression is identically equal to x_5^3. Then, in order to maximize it, we must minimize $x_1+x_2+x_3+x_4$. These are distinct positive integers, so their sum will be smallest when we have $\{x_1, x_2, x_3, x_4\} = \{1, 2, 3, 4\}$, giving $x_5 \leq 90$. This gives us our maximum of $90^3 = \boxed{729000}$.

Comment: This problem is somewhat related to computing the contour integral of $\frac{1}{q(z)}$ for polynomial $q(z)$ with distinct (not necessarily real) roots over a contour containing all these roots. If you are interested, explore in that direction and you will find interesting results.