1. Let $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$. Compute $f'(2)$.

   Answer: 405

   Solution 1: We directly compute $f'(x) = 5x^4 + 20x^3 + 30x^2 + 20x + 5$. Plugging in $x = 2$, we get $\boxed{405}$.

   Solution 2: We first factor $f(x) = (x+1)^5$. Then $f'(x) = 5(x+1)^4$ and plugging in $x = 2$, we get $\boxed{405}$.

2. There are 5 contestants in the Rice Marathon Tournament, numbered 1 to 5. After t hours, contestant n has run $\frac{t^n}{2}$ miles. Compute the average speed of the 5 contestants at time $t = 2$, in miles per hour.

   Answer: $\frac{129}{10}$ or 12.9

   Solution 1: The speed of contestant n at time $t = 2$ is $\frac{d}{dt}\left(\frac{t^n}{2}\right)\Big|_{t=2} = \frac{n2^{n-1}}{2} = n2^{n-2}$. Thus, the sum of the contestants' speeds at time $t = 2$ is $\frac{1}{2} + 2\cdot 1 + 3\cdot 2 + 4\cdot 4 + 5\cdot 8 = \frac{129}{2}$. The average speed is therefore $\boxed{\dfrac{129}{10}}$.

   Solution 2: Let $x_n(t) = t^n$. Then the average speed of the 5 contestants is given by

$$\begin{aligned}\frac{x_1'(2) + x_2'(2) + \cdots + x_5'(2)}{5} &= \frac{d}{dt}\left(\frac{(x_1(t) + \cdots + x_5(t))}{5}\right)\Big|_{t=2} \\ &= \frac{1}{10}\frac{d}{dt}(t + t^2 + \cdots + t^5)\Big|_{t=2} \\ &= \frac{1}{10}\frac{d}{dt}\left(\frac{t^6 - t}{t-1}\right)\Big|_{t=2} \\ &= \frac{1}{10}\frac{(6t^5-1)(t-1) - (t^6 - t)}{(t-1)^2}\Big|_{t=2} \\ &= \boxed{\frac{129}{10}}.\end{aligned}$$

3. Moor is trying to paint the interval $[0, 5]$ using red and green paints. Painting at the point x using red paint costs $2x$ dollars per unit length and using green paint costs x^2 dollars per unit length. What is the minimum amount of money Moor needs to spend to paint the entire interval if he's allowed to change colors as he paints?

   Answer: $\frac{71}{3}$

   Solution: For $0 \le x \le 2$, it is cheaper for Moor to use green paint since $x^2 \le 2x$ in the interval $[0, 2]$. For the interval $[2, 5]$, it is cheaper for Moor to use red paint. Thus, the minimum amount of money Moor needs to spend is

$$\int_0^2 x^2\,dx + \int_2^5 2x\,dx = \boxed{\frac{71}{3}}.$$

4. Let $y(u)$ be the largest of the roots of $x^2 + ux - 7 = 0$. If u is increasing by 2 per second, what is the rate of change of $y(u)$ when $y(u) = 4$?

Answer: $-\frac{32}{23}$

Solution: Let $z(u)$ be the smaller root of the polynomial. Then $y(u)z(u) = -7$ and $y(u)+z(u) = -u$, so $y(u) - 7/y(u) = -u$. Using differentiating by t gives

$$\frac{dy}{dt} + \frac{7}{y^2}\frac{dy}{dt} = -\frac{du}{dt}.$$

Plugging in the given quantities yields the answer $\boxed{-\frac{32}{23}}$.

5. Given that α and β are positive real numbers, compute the following limit (where it exists and is nonzero) in terms of α and β:

$$\lim_{x\to 0^+} \frac{\sin x^\alpha}{\cos x^\beta - 1}.$$

Answer: -2

Solution 1: Since $\alpha, \beta > 0$, we may apply l'Hopital's rule so that the given limit is equal to $-\frac{\alpha}{\beta}\lim_{x\to 0}\frac{x^{\alpha-\beta}\cos x^\alpha}{\sin x^\beta}$. Noting that $\cos(0) = 1$ and that the existence of our limit is assumed, we can simplify the expression to just $-\frac{\alpha}{\beta}\lim_{x\to 0}\frac{x^{\alpha-\beta}}{\sin x^\beta}$. Then: $-\frac{\alpha}{\beta}\lim_{x\to 0}\frac{x^{\alpha-\beta}}{\sin x^\beta} = -\frac{\alpha}{\beta}\lim_{x\to 0}\frac{x^\beta}{\sin x^\beta}\cdot x^{\alpha-2\beta} = -\frac{\alpha}{\beta}\lim_{x\to 0} x^{\alpha-2\beta}$. Since the limit is nonzero, we must have $\alpha = 2\beta$ giving us the answer $\lim_{x\to 0}\frac{\sin x^\alpha}{\cos x^\beta - 1} = \boxed{-2}$.

Solution 2: This problem may easily be solved by using Taylor series expansions, which are very well known for sine and cosine. Specifically,

$$\lim_{x\to 0}\frac{\sin x^\alpha}{\cos x^\beta - 1} = \lim_{x\to 0}\frac{x^\alpha - \frac{x^{3\alpha}}{6} + \cdots}{\left(1 - \frac{x^{2\beta}}{2} + \cdots\right) - 1} = \lim_{x\to 0} x^{\alpha-2\beta}\left[\frac{1 - \frac{x^{2\alpha}}{6} + \cdots}{-\frac{1}{2} + \frac{x^{2\beta}}{24} - \cdots}\right].$$

Because the limit is nonzero, we must have $\alpha = 2\beta$, and then plugging $x = 0$ into the last expression yields the answer $\lim_{x\to 0}\frac{\sin x^\alpha}{\cos x^\beta - 1} = \frac{1}{-\frac{1}{2}} = \boxed{-2}$.

6. Compute

$$\frac{d}{dx}\prod_{n=1}^{16}\left(x + \frac{1}{n}\right)\Bigg|_{x=0}.$$

Answer: $\frac{17}{2\times 15!}$ **or** $\frac{136}{16!}$

Solution: Using the product rule,

$$\frac{d}{dx}\prod_{n=1}^{16}\left(x + \frac{1}{n}\right)\Bigg|_{x=0} = \sum_{k=1}^{16}\frac{\prod_{n=1}^{16}\left(x + \frac{1}{n}\right)}{x + \frac{1}{k}}\Bigg|_{x=0} = \sum_{k=1}^{16}\frac{\prod_{n=1}^{16}\frac{1}{n}}{\frac{1}{k}} = \sum_{k=1}^{16}\frac{k}{16!} = \boxed{\frac{17}{2\times 15!}}.$$

7. Compute

$$\int_0^2 \sqrt{(2-x)\left(\sqrt{x}+\sqrt{x+2}\right)^2}\,dx.$$

Answer: $\mathbf{\frac{3\pi}{2}}$

Solution: Rewrite the integral as

$$\int_0^2 \sqrt{2-x}\left(\sqrt{x}+\sqrt{x+2}\right)\,dx = \int_0^2 \sqrt{2x-x^2}\,dx + \int_0^2 \sqrt{4-x^2}\,dx.$$

Each integral gives the area of part of a circle. The first integral is the area of a semicircle with radius 1, centered at $(1,0)$; the second integral is the area of a quarter circle with radius 2 centered at the origin. Therefore, the answer is the answer is $\frac{1}{2}\pi + \frac{1}{4}(4\pi) = \boxed{\frac{3\pi}{2}}$.

8. Compute

$$2014\int_0^\infty \frac{(1+x)^{2013}}{(2+x)^{2015}}\,dx.$$

Answer: $\mathbf{1-\frac{1}{2^{2014}}}$ **or** $\mathbf{1-2^{-2014}}$

Solution: First make the substitution $y = 2+x$. This gives

$$\int_0^\infty \frac{(1+x)^{2013}}{(2+x)^{2015}}\,dx = \int_2^\infty \frac{(y-1)^{2013}}{y^{2015}}\,dy = \int_2^\infty \frac{1}{y^2}\left(\frac{y-1}{y}\right)^{2013}\,dy = \int_2^\infty \frac{1}{y^2}\left(1-\frac{1}{y}\right)^{2013}\,dy.$$

Now, we make the substitution $u = 1-\frac{1}{y}$, so that $du = \frac{1}{y^2}\,dy$. Then we have

$$= \int_{\frac{1}{2}}^1 u^{2013}\,du = \left.\frac{u^{2014}}{2014}\right|_{\frac{1}{2}}^1 = \frac{1}{2014}\left(1-\left(\frac{1}{2}\right)^{2014}\right)$$

So our answer is

$$2014\cdot\int_0^\infty \frac{(1+x)^{2013}}{(2+x)^{2015}}\,dx = \boxed{1-2^{-2014}}$$

9. Given that it converges, compute the following infinite product:

$$\prod_{n=1}^\infty \frac{5^{2^{-n}}+3^{2^{-n}}}{2}.$$

Answer: $\mathbf{\frac{2}{\ln\left(\frac{5}{3}\right)}}$ **or** $\mathbf{\frac{2}{\ln(5)-\ln(3)}}$

Solution 1: Let x be a random variable uniformly distributed on the interval $[0,1]$. The infinite product we wish to calculate is the expected value of $5^x 3^{1-x}$.

To see this, consider the infinite product

$$\left(\frac{5^{2^{-1}}}{2}+\frac{3^{2^{-1}}}{2}\right)\left(\frac{5^{2^{-2}}}{2}+\frac{3^{2^{-2}}}{2}\right)\cdots.$$

For each term $\frac{5^{2^{-n}}}{2} + \frac{3^{2^{-n}}}{2}$, we may choose either $5^{2^{-n}}$ or $3^{2^{-n}}$ depending on whether the nth digit of the binary expansion of x is a 1 or 0. Then the product will be equal to $5^x 3^{1-x}$. We divide by 2 on each choice since x is uniformly distributed.

Thus, the infinite product equals the expected value of $5^x 3^{1-x}$ where x is uniformly distributed on the interval $[0, 1]$. This is equal to the integral

$$\begin{aligned}\int_0^1 5^x 3^{1-x}\,dx &= 3\int_0^1 \left(\frac{5}{3}\right)^x dx \\ &= 3\int_0^1 e^{x\ln\frac{5}{3}}\,dx \\ &= 3\cdot\frac{1}{\ln\left(\frac{5}{3}\right)}\cdot\left(\frac{5}{3}-1\right) \\ &= \boxed{\frac{2}{\ln\left(\frac{5}{3}\right)}}.\end{aligned}$$

Solution 2: Let

$$P_n = \prod_{k=1}^{n} \frac{5^{2^{-k}} + 3^{2^{-k}}}{2} = \frac{1}{2^n}\prod_{k=1}^{n}\left(5^{2^{-k}} + 3^{2^{-k}}\right).$$

Observe that $2^n\left(5^{2^{-n}} - 3^{2^{-n}}\right) P_n = 2$ for all n, as the series telescopes. In order to compute the limit

$$\lim_{n\to\infty} 2^n\left(5^{2^{-n}} - 3^{2^{-n}}\right),$$

we remove the restriction to the naturals and let the limit to infinity be realized along the reals – note that if this limit exists, then the limit of the sequence must also exist and be the same. We compute

$$\begin{aligned}\lim_{x\to\infty} 2^x\left(5^{2^{-x}} - 3^{2^{-x}}\right) &= \lim_{x\to\infty}\frac{5^{2^{-x}} - 3^{2^{-x}}}{2^{-x}} = \lim_{x\to\infty}\frac{(2^{-x})'\left(\ln(5)5^{2^{-x}} - \ln(3)3^{2^{-x}}\right)}{(2^{-x})'} \\ &= \lim_{x\to\infty}\ln(5)5^{2^{-x}} - \ln(3)3^{2^{-x}} = \ln(5) - \ln(3) = \ln\left(\frac{5}{3}\right).\end{aligned}$$

Thus,

$$P_n = \frac{2}{2^n\left(5^{2^{-n}} - 3^{2^{-n}}\right)} \to \boxed{\frac{2}{\ln\left(\frac{5}{3}\right)}} \text{ as } n\to\infty.$$

10. Consider the real-valued differential equation $u''(x) = u^2(x) - u^5(x)$. Suppose that $u'(0) = 7$ and $u(0) = 2$. Compute the maximum possible value of $|u'(x)|$.

Answer: $\mathbf{\frac{14\sqrt{3}}{3}}$

Solution: First, multiply both sides of the equation by u' and anti-differentiate to obtain

$$0 = u'(u'' - u^2 + u^5) = \frac{d}{dx}\left(\frac{(u')^2}{2} - \frac{u^3}{3} + \frac{u^6}{6}\right).$$

This implies that because the derivative is zero, the quantity in the brackets on the right hand side must be constant for all x; let this constant be

$$K = \frac{(u')^2}{2} - \frac{u^3}{3} + \frac{u^6}{6}.$$

At $x = 0$, we have $u = 2$ and $u' = 7$; this means that we have $K = \frac{49}{2} - \frac{2^3}{3} + \frac{2^6}{6} = \frac{65}{2}$. We can rearrange to see that $(u')^2 = 2K + \frac{2u^3}{3} - \frac{u^6}{3}$.

Observe that at the maximum of $|u'|$, we must have $0 = u'' = u^2 - u^5$. Therefore, the maximum value of $|u'|$ is obtained at a solution of $u^2 - u^5 = 0$, i.e. $u = 0$ or $u = 1$. Checking both possibilities, the value of $|u'|$ when $u = 1$ is greater. Therefore, the maximum is

$$u' = \sqrt{2K + \frac{2u^3}{3} - \frac{u^6}{3}} = \sqrt{2 \cdot \frac{65}{2} + \frac{2}{3} - \frac{1}{3}} = \boxed{\frac{14\sqrt{3}}{3}}.$$

For purposes of rigor, we should argue that there exists x such that $u(x) = 1$. Define $v = u^3$ and $w = u'$. Then it is clear that the solutions u satisfy $w^2 = 65 + 2/3v - v^2/6$. This is an ellipse in the w-v plane that clearly intersects the line $v = 1$. Now, the solutions u must correspond to continuous curves that are a subset of this ellipse. To show the desired result, it suffices to show that u covers the whole ellipse.

In the new variables, the ODE has phase curves in $\mathbb{R}^2$ according to the equation $(w', u') = (u^2 - u^5, w)$. By the uniqueness of solutions to ODEs, the point given in the problem defines a unique phase curve on which the solution lies. It must also satisfy the equation of the ellipse. Finally, it covers the entire ellipse because the following vector field is bounded away from 0 on the ellipse:

$$(w', u') = \left(u^2 - u^5, \sqrt{2\left(\frac{65}{2} + \frac{u^3}{3} - \frac{u^6}{6}\right)}\right) \neq 0.$$

Therefore, the flow cannot approach a fixed point (or limit point) on the ellipse, so it covers the ellipse.