1. In a Super Smash Brothers tournament, $\frac{1}{2}$ of the contestants play as Fox, $\frac{1}{3}$ of the contestants play as Falco, and $\frac{1}{6}$ of the contestants play as Peach. Given that there were 40 more people who played either Fox or Falco than who played Peach, how many contestants attended the tournament?

   Answer: 60

   Solution: Let x denote the number of contestants in the tournament. Then $\frac{1}{2}x + \frac{1}{3}x - \frac{1}{6}x = 40$. Thus, $\frac{2}{3}x = 40$ and hence $x = \boxed{60}$ contestants attended the tournament.

2. Find all pairs (x, y) that satisfy

$$x^2 + y^2 = 1$$
$$x + 2y = 2$$

   Answer: $(\mathbf{0}, \mathbf{1})$ and $(\frac{4}{5}, \frac{3}{5})$

   Solution: The second equation tells us that $x = 2-2y$. Substituting this into the first equation, we have

$$(2 - 2y)^2 + y^2 = 1$$
$$5y^2 - 8y + 3 = 0$$
$$(5y - 3)(y - 1) = 0$$

   Thus, $y = \frac{3}{5}$ or $y = 1$. Plugging in these values of y and solving for x, we find that the possible solutions to the system are $\boxed{(0, 1), \left(\frac{4}{5}, \frac{3}{5}\right)}$.

3. Find the unique $x > 0$ such that $\sqrt{x} + \sqrt{x + \sqrt{x}} = 1$.

   Answer: $\frac{1}{9}$

   Solution: We solve

$$\sqrt{x} + \sqrt{x + \sqrt{x}} = 1$$
$$\sqrt{x + \sqrt{x}} = 1 - \sqrt{x}$$
$$x + \sqrt{x} = (1 - \sqrt{x})^2$$
$$x + \sqrt{x} = 1 + x - 2\sqrt{x}$$
$$3\sqrt{x} = 1$$
$$x = \boxed{\frac{1}{9}}$$

4. Find the sum of all real roots of $x^5 + 4x^4 + x^3 - x^2 - 4x - 1$.

   Answer: -3

Solution 1: Notice that $x^5 + 4x^4 + x^3 - x^2 - 4x - 1 = x^3(x^2 + 4x + 1) - (x^2 + 4x + 1) = (x^3 - 1)(x^2 + 4x + 1)$. The only real root of $x^3 - 1$ is 1, and the real roots of $x^2 + 4x + 1$ are $-2 \pm \sqrt{3}$ by the quadratic formula. Thus, the sum of all real roots of the polynomial is $1 + (-2 + \sqrt{3}) + (-2 - \sqrt{3}) = \boxed{-3}$.

Solution 2: By the rational root theorem, we quickly discover that 1 is a root of the quintic polynomial. Factoring $x - 1$ out, we are left with the quartic $x^4 + 5x^3 + 6x^2 + 5x + 1$. This polynomial is symmetric, so we may factor it as $x^2((x + \frac{1}{x})^2 + 5(x + \frac{1}{x}) + 4)$. Thus, it suffices to find the roots of $(x + \frac{1}{x})^2 + 5(x + \frac{1}{x}) + 4$. Writing $y = x + \frac{1}{x}$, we have $y^2 + 5y + 4$ which has roots $y = -1, -4$. Thus, the roots of the quartic are the solutions to the equations $x + \frac{1}{x} = -1$ and $x + \frac{1}{x} = -4$. Solving the first equation, we have

$$\begin{aligned} x + \frac{1}{x} &= -1 \\ x^2 + x + 1 &= 0 \end{aligned}$$

which has no real roots. The second equation gives

$$\begin{aligned} x + \frac{1}{x} &= -4 \\ x^2 + 4x + 1 &= 0 \end{aligned}$$

which has the roots $x = -2 \pm \sqrt{3}$.

Therefore, the sum of all real roots of the polynomial is $1 + (-2 + \sqrt{3}) + (-2 - \sqrt{3}) = \boxed{-3}$.

5. Let $f(a, b) = \frac{1}{a+b}$ when $a + b \neq 0$. Suppose that x, y, z are distinct integers such that $x + y + z = 2015$ and $f(f(x, y), z) = f(x, f(y, z))$ (where both sides of the equation exist and are well-defined). Compute y.

   Answer: -2015

   Solution: Since $f(f(x, y), z) = f(x, f(y, z))$, we have that

$$\frac{1}{\frac{1}{x+y} + z} = \frac{1}{x + \frac{1}{y+z}}$$

Simplifying, we have that

$$\begin{aligned} x + \frac{1}{y+z} &= \frac{1}{x+y} + z \\ x(y+z)(x+y) + (x+y) &= (y+z) + z(x+y)(y+z) \\ x\left((y+z)(x+y) + 1\right) &= z\left((x+y)(y+z) + 1\right) \\ (x-z)\left((y+z)(x+y) + 1\right) &= 0 \end{aligned}$$

Since x and z are distinct, $x - z \neq 0$ so we may divide through by $x - z$ to obtain

$$\begin{aligned} (y+z)(x+y) + 1 &= 0 \\ (y+z)(x+y) &= -1 \end{aligned}$$

Since $x + y + z = 2015$, $y + z = 2015 - x$ and $x + y = 2015 - z$ so

$$(2015 - x)(2015 - z) = -1$$

Since x, y, z are integers, we have that $x = 2014$ and $z = 2016$ (or the other way around). In either case, $y = 2015 - x - z = 2015 - 2014 - 2016 = \boxed{-2015}$.

6. Compute all pairs of real numbers (a, b) such that the polynomial $f(x) = (x^2 + ax + b)^2 + a(x^2 + ax + b) - b$ has exactly one real root and no complex roots.

Answer: $\mathbf{(0, 0), \left(1, -\frac{1}{4}\right)}$

Solution: Let $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + ax - b$. Then we are looking for a, b such that $Q(P(x))$ has only a single real repeated root. The roots of $Q(P(x))$ are the solutions to the equations $P(x) = r_1$ and $P(x) = r_2$ where r_1, r_2 are the roots of Q. Thus, a necessary condition is for $Q(x)$ to have a repeated root, so we require that the discriminant $a^2 + 4b = 0$. Then the repeated root of Q is $r = -\frac{a}{2}$.

Now, we require that $P(x) = -\frac{a}{2}$ have a repeated root, so the discriminant of $P(x) + \frac{a}{2}$ must be zero. Therefore, we require that $a^2 - 4(b + \frac{a}{2}) = 0$. From earlier, we know that $a^2 + 4b = 0$, so we may substitute in $b = -\frac{a^2}{4}$. Hence, we have the equation

$$\begin{aligned} a^2 - 4\left(-\frac{a^2}{4} + \frac{a}{2}\right) &= 0 \\ a^2 - a &= 0 \\ a(a - 1) &= 0 \end{aligned}$$

So $a = 0, 1$ with corresponding $b = 0, -\frac{1}{4}$. Thus, the desired pairs of a and b are $\boxed{(0, 0), \left(1, -\frac{1}{4}\right)}$.

7. Let a, b, c, d be real numbers that satisfy

$$\begin{aligned} ab + cd &= 11 \\ ac + bd &= 13 \\ ad + bc &= 17 \\ abcd &= 30 \end{aligned}$$

Find the greatest possible value of a.

Answer: $\mathbf{\sqrt{30}}$

Solution: Note that ab, cd are roots of the quadratic equation $x^2 - 11x + 30$ because $ab + cd = 11$ and $ab \cdot cd = abcd = 30$. But this clearly has roots $5, 6$, thus $\{ab, cd\} = \{5, 6\}$. Similarly, we must have that

$$\{ab, cd\} = \{5, 6\}, \quad \{ac, bd\} = \{3, 10\}, \quad \{ad, bc\} = \{2, 15\}.$$

But we have that $a^2 = \frac{ab \cdot ac \cdot ad}{abcd}$ is maximized when we maximize ab, ac, ad. Using our previous result, we set $(ab, cd) = (6, 5), (ac, bd) = (10, 3), (ad, bc) = (15, 2)$ and conclude that the maximum of a^2 is $\frac{6 \cdot 10 \cdot 15}{30} = 30$, thus $a \leq \sqrt{30}$. Note that $(a, b, c, d) = \left(\sqrt{30}, \sqrt{\frac{6}{5}}, \sqrt{\frac{10}{3}}, \sqrt{\frac{15}{2}}\right)$ satisfies the equation (b^2, c^2, d^2 were found in analogous ways to a^2), thus we have $a = \boxed{\sqrt{30}}$ is the greatest possible value of a.

8. The polynomial $x^7 + x^6 + x^4 + x^3 + x + 1$ has roots $r_1, r_2, r_3, r_4, r_5, r_6, r_7$. Calculate

$$\sum_{n=1}^{7} r_n^3 + \frac{1}{r_n^3}$$

Answer: -8

Solution 1: First, notice that -1 is a root of the polynomial. Factoring, we obtain $x^7 + x^6 + x^4 + x^3 + x + 1 = (x+1)(x^6 + x^3 + 1)$. Now, since $(x^6 + x^3 + 1)(x^3 - 1) = x^9 - 1$, it follows that the roots of $x^6 + x^3 + 1$ are the ninth primitive roots of unity. That is, the roots are $x = e^{\frac{2\pi ki}{9}}$ for integer k that are coprime with 9. So the roots of x^6+x^3+1 are $e^{\frac{2\pi i}{9}}, e^{\frac{4\pi i}{9}}, e^{\frac{8\pi i}{9}}, e^{\frac{10\pi i}{9}}, e^{\frac{14\pi i}{9}}, e^{\frac{16\pi i}{9}}$.
Now, since $\left(e^{\frac{2\pi ki}{9}}\right)^3 = e^{\frac{2\pi ki}{3}}$ and $\left(e^{\frac{2\pi ki}{9}}\right)^{-3} = e^{-\frac{2\pi k}{3}}$. Next, since $e^{i\theta} = \cos\theta + i\sin\theta$, it follows that

$$e^{\frac{2\pi ki}{3}} + e^{-\frac{2\pi ki}{3}} = 2\cos\left(\frac{2\pi k}{3}\right)$$

In addition, for the root -1, we have $(-1)^3 + (-1)^{-3} = -2$. Therefore, the sum we seek is equal to

$$-2 + 2\cos\left(\frac{2\pi}{3}\right) + 2\cos\left(\frac{4\pi}{3}\right) + 2\cos\left(\frac{8\pi}{3}\right) + 2\cos\left(\frac{10\pi}{3}\right) + 2\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{16\pi}{3}\right) = \boxed{-8}$$

Solution 2: We first make the observation that if r is a root of the polynomial, then so is r^{-1}. To see this, suppose r is a root of the polynomial. Then $r^7+r^6+r^4+r^3+r+1 = 0$. Dividing through by r^7 (which is nonzero since 0 is not a root), we find that $1 + r^{-1} + r^{-3} + r^{-4} + r^{-6} + r^{-7} = 0$ so r^{-1} is also a root.

Therefore it follows that $\sum_{n=1}^{7} r_n^3 = \sum_{n=1}^{7} r_n^{-3}$ so the sum desired is precisely $2 \cdot \sum_{n=1}^{7} r_n^3$. We now proceed to compute $\sum_{n=1}^{7} r_n^3$ via Newton's identities. Let a_k denote the coefficient of x^k in the polynomial and let P_k denote the sum $\sum_{n=1}^{7} r_n^k$. Then Newton's identities gives us the following set of equations:

$$a_7P_1 + a_6 = 0$$
$$a_7P_2 + a_6P_1 + 2a_5 = 0$$
$$a_7P_3 + a_6P_2 + a_5P_1 + 3a_4 = 0$$

Substituting in $a_7 = a_6 = a_4 = 1$ and $a_5 = 0$, we get the system

$$P_1 + 1 = 0$$
$$P_2 + P_1 = 0$$
$$P_3 + P_2 + 3 = 0$$

which solves to $P_1 = -1$, $P_2 = 1$, and $P_3 = -4$. Thus, $P_3 = \sum n = 1^7 r_n^3 = -4$ and hence the desired sum is $2 \cdot -4 = \boxed{-8}$.

9. Given that real numbers x, y satisfy the equation $x^4 + x^2y^2 + y^4 = 72$, what is the minimum possible value of $2x^2 + xy + 2y^2$?

Answer: $6\sqrt{6}$

Solution 1: Let $a = x^2 + xy + y^2$ and $b = x^2 - xy + y^2$. Then $ab = x^4 + x^2y^2 + y^4 = 72$ and the expression we are trying to minimize is $\frac{a+b}{2} + a$. Substituting $b = \frac{72}{a}$, the expression we want to minimize becomes $\frac{3a}{2} + \frac{36}{a}$. By AM-GM, this is greater than or equal to $2\sqrt{\frac{3a}{2} \cdot \frac{36}{a}} = 2\sqrt{54} = 6\sqrt{6}$.

Finally, we show $6\sqrt{6}$ is achievable. In AM-GM, equality is only achieved when $\frac{3a}{2} = \frac{36}{a}$ so $a = 2\sqrt{6}$. If we let $x = \sqrt{2\sqrt{6}}$ and $y = -\sqrt{2\sqrt{6}}$, then $a = 2\sqrt{6}$ so $6\sqrt{6}$ is achievable. Hence, the minimum possible value is $\boxed{6\sqrt{6}}$.

Solution 2: Let $a = x^2+y^2$ and $b = xy$. Then we are trying to minimize $2a+b$. The expression $x^4+x^2y^2+y^4$ can be factored as $(x^2+y^2)^2-(xy)^2 = (a-b)(a+b)$. Thus, we have the condition $(a-b)(a+b) = 72$. Now, notice that $|x^2+y^2| > |xy|$ for any choice of x, y, so both $a-b$ and $a+b$ are always positive.

Now, suppose $a - b = r$. Then the condition $(a-b)(a+b) = 72$ tells us that $a+b = \frac{72}{r}$. Hence, we can solve the system of equations

$$a - b = r$$
$$a + b = \frac{72}{r}$$

to obtain $a = \frac{36}{r} + \frac{r}{2}$ and $b = \frac{36}{r} - \frac{r}{2}$. The expression we are trying to minimize is thus $2a + b = \frac{108}{r} + \frac{r}{2}$. By AM-GM, this is greater than or equal to $2\sqrt{\frac{108}{r} \cdot \frac{r}{2}} = 6\sqrt{6}$.

Finally, we show that this is achievable. Equality holds in AM-GM precisely when $\frac{108}{r} = \frac{r}{2}$ or when $r = 6\sqrt{6}$. Then $a = 4\sqrt{6}$ and $b = -2\sqrt{6}$. So this is achievable by finding x, y such that $x^2 + y^2 = 4\sqrt{6}$ and $xy = -2\sqrt{6}$. One such x, y is $x = \sqrt{2\sqrt{6}}$ and $y = -\sqrt{2\sqrt{6}}$. Thus, the minimum possible value is indeed $\boxed{6\sqrt{6}}$.

10. Consider a sequence defined recursively by $a_n = 1 + (a_0+1)(a_1+1)\cdots(a_{n-1}+1)$. Let $-2 < a_0 < -1$ such that

$$\sum_{n=0}^{2015} \frac{a_n}{a_n^2-1} = -\frac{a_0+4}{a_0^2-1}.$$

What is the value of a_0?

Answer: $\mathbf{3^{-\frac{1}{2^{2015}-1}} - 2}$ **or** $\mathbf{\frac{1}{3^{\frac{1}{2^{2015}-1}}} - 2}$

Solution: First, notice that for $n > 1$,

$$\begin{aligned} a_n &= 1 + (a_0+1)(a_1+1)\cdots(a_{n-2}+1)(a_{n-1}+1) \\ &= 1 + (a_{n-1}-1)(a_{n-1}+1) \\ &= a_{n-1}^2 \end{aligned}$$

and that $a_1 = a_0 + 2$. Therefore, for $n \geq 1$, $a_n = (a_0+2)^{2^{n-1}}$.

Next, notice that for $n > 0$

$$\begin{aligned} \frac{1}{a_n-1} - \frac{1}{a_{n+1}-1} &= \frac{1}{(a_0+1)(a_1+1)\cdots(a_{n-1}+1)} - \frac{1}{(a_0+1)(a_1+1)\cdots(a_n+1)} \\ &= \frac{a_n}{(a_0+1)\cdots(a_n+1)} \\ &= \frac{a_n}{(a_n-1)(a_n+1)} \\ &= \frac{a_n}{a_n^2-1} \end{aligned}$$

Therefore, our sum telescopes as follows

$$\begin{aligned}\frac{a_0}{a_0^2-1}+\sum_{n=1}^{2015}\frac{a_n}{a_n^2-1} &= \frac{a_0}{a_0^2-1}+\sum_{n=1}^{2015}\left(\frac{1}{a_n-1}-\frac{1}{a_{n+1}-1}\right)\\ &= \frac{a_0}{a_0^2-1}+\frac{1}{a_1-1}-\frac{1}{a_{2016}-1}\end{aligned}$$

Using the fact that $a_1 = a_0 + 2$ and $a_{2016} = (a_0+2)^{2^{2015}}$, we can simplify the above expression to

$$\frac{a_0}{a_0^2-1}+\frac{1}{a_0+1}-\frac{1}{(a_0+2)^{2^{2015}}-1}$$

If we let $x = a_0 + 2$, then the condition in the problem statement becomes

$$\begin{aligned}\frac{x-2}{(x-2)^2-1}+\frac{1}{x-1}-\frac{1}{x^{2^{2015}}-1} &= -\frac{x+2}{(x-2)^2-1}\\ \frac{2x}{(x-1)(x-3)}+\frac{1}{x-1}-\frac{1}{x^{2^{2015}}-1} &= 0\\ \frac{3x-3}{(x-1)(x-3)}-\frac{1}{x^{2^{2015}}-1} &= 0\\ \frac{3}{x-3}-\frac{1}{x^{2^{2015}}-1} &= 0\\ 3(x^{2^{2015}}-1)-(x-3) &= 0\\ 3x^{2^{2015}}-x &= 0\end{aligned}$$

Since we want $-2 < a_0$, $0 < x$, so we can divide through by x to get

$$\begin{aligned}3x^{2^{2015}-1}-1 &= 0\\ x &= 3^{-\frac{1}{2^{2015}-1}}\end{aligned}$$

And hence $a_0 = \boxed{3^{-\frac{1}{2^{2015}-1}}-2}$.