1. Given that the three points where the parabola $y = bx^2 - 2$ intersects the x-axis and y-axis form an equilateral triangle, compute b.

   Answer: $\mathbf{\frac{3}{2}}$

   Solution: Note that the three points are $(-a, 0)$ and $(a, 0)$ for some a, and $(0, -2)$. We therefore have that $2a = \sqrt{a^2 + 4}$, so $a = \sqrt{\frac{4}{3}}$, meaning that $\frac{4b}{3} - 2 = 0$ so $\boxed{b = \frac{3}{2}}$.

2. Compute the last digit of $2^{\left(3^{\left(4^{\cdots 2014}\right)}\right)}$.

   Answer: 2

   Solution: The exponent of 2 is equivalent to 1 (mod 4). Since $2^x \pmod{10}$ has period 4, we have that $2^1 \equiv \boxed{2} \pmod{10}$.

3. A math tournament has a test which contains 10 questions, each of which come from one of three different subjects. The subject of each question is chosen uniformly at random from the three subjects, and independently of the subjects of all the other questions. The test is *unfair* if any one subject appears at least 5 times. Compute the probability that the test is unfair.

   Answer: $\mathbf{\frac{4111}{6561}}$

   Solution: A fair (not unfair) test can either have 4, 3, and 3 questions in each subject, or 4, 4, and 2 questions. Thus there are $3 \cdot \binom{10}{4} \cdot \binom{6}{3} + 3 \cdot \binom{10}{4} \cdot \binom{6}{4} = 22050$ distinct fair tests. This means the probability that the test is fair is $\frac{22050}{3^{10}} = \frac{2450}{6561}$, so the probability that the test is unfair is $1 - \frac{2450}{6561} = \boxed{\frac{4111}{6561}}$.

4. Let S_n be the sum $S_n = 1 + 11 + 111 + 1111 + \ldots + 111\ldots11$ where the last number $111\ldots11$ has exactly n 1's. Find $\lfloor 10^{2017}/S_{2014} \rfloor$.

   Answer: 8100

   Solution: First we want to find an explicit formula for S_n. This is not too difficult: noting that $111\ldots11 = \frac{10^n-1}{9}$, our sum is equal to

$$S_n = \sum_{i=1}^{n} \frac{10^i - 1}{9} = \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1\right) = \frac{1}{9}\left(\frac{10^{n+1} - 1}{9} - n\right) = \frac{10^{n+1} - 1 - 9n}{81}.$$

   So, $\frac{10^{2017}}{S_{2014}} = \frac{81 \cdot 10^{2017}}{10^{2015} - 1 - 9 \cdot 2014}$ is just a tiny bit larger than $\frac{81 \cdot 10^{2017}}{10^{2015}} = 8100$. So, the answer is $\boxed{8100}$.

5. ABC is an equilateral triangle with side length 12. Let O_A be the point inside ABC that is equidistant from B and C and is $\sqrt{3}$ units from A. Define O_B and O_C symmetrically. Find the area of the intersection of triangles O_ABC, AO_BC, and ABO_C.

   Answer: $\mathbf{\frac{162\sqrt{3}}{7}}$

   Solution: Let (ABC) denote the area of the polygon ABC. The Principle of Inclusion-Exclusion, along with the symmetry between O_ABC, AO_BC, and ABO_C tells us that

$$(O_ABC \cup AO_BC \cup ABO_C) = 3(O_ABC) - 3(O_ABC \cap AO_BC) + (O_ABC \cap AO_BC \cap ABO_C).$$

$O_ABC \cup AO_BC \cup ABO_C$ is simply ABC, whose area is $\frac{12^2\sqrt{3}}{4} = 36\sqrt{3}$. The area of O_ABC is also easy to compute. The altitude from A to BC goes through O_A, since A and O_A are both on the perpendicular bisector of BC. Since the altitude from A to BC has length $6\sqrt{3}$, O_ABC is a triangle with height $5\sqrt{3}$ and base 12, and hence has area $30\sqrt{3}$.

Now, we calculate $(O_ABC \cap AO_BC)$. Let O be the orthocenter of ABC. As we showed earlier, O_A is on AO and O_B is on BO. Let AO_B and BO_A intersect at D. It is easy to see that $AD \cong BD$, so D lies on the perpendicular bisector of AB, which is CO. We can also see without too much work that $O_ABC \cap AO_BC = CO_ADO_B$, and triangles CO_AD and CO_BD are congruent, so we just need to find the area of CO_AD and multiply by two.

Since $O_ABC = O_ACD \cup DCB$, it suffices to find the area of DCB. Properties of medians or direct computation with 30-60-90 triangles tells us that $AO = BO = 4\sqrt{3}$. Since OD bisects $\angle O_AOB$, we have

$$\begin{aligned}\frac{BD}{BO} = \frac{O_AD}{O_AO} &\implies \frac{BD}{4\sqrt{3}} = \frac{O_AD}{3\sqrt{3}} \\ &\implies \frac{O_AD}{BD} = \frac{3}{4} \\ &\implies \frac{O_AD}{O_AB} = \frac{3}{7}.\end{aligned}$$

Now, we have that

$$(CO_AD) = \frac{3}{7}(CO_AB) = \frac{3}{7} \cdot 30\sqrt{3} = \frac{90\sqrt{3}}{7} \implies (CO_ADO_B) = \frac{180\sqrt{3}}{7}.$$

Finally, plugging into PIE, we get

$$(O_ABC \cap AO_BC \cap ABO_C) = 36\sqrt{3} - 3 \cdot 30\sqrt{3} + 3 \cdot \frac{180\sqrt{3}}{7} = \frac{540 - 54 \cdot 7}{7}\sqrt{3} = \boxed{\frac{162\sqrt{3}}{7}}.$$

6. A composition of a natural number n is a way of writing it as a sum of natural numbers, such as $3 = 1 + 2$. Let $P(n)$ denote the sum over all compositions of n of the number of terms in the composition. For example, the compositions of 3 are 3, $1+2$, $2+1$, and $1+1+1$; the first has one term, the second and third have two each, and the last has 3 terms, so $P(3) = 1+2+2+3 = 8$. Compute $P(9)$.

Answer: 1280

Solution: First, for $1 \le k \le n$, the number of compositions with k parts is $\binom{n-1}{k-1}$. This is because every composition can be described uniquely by collapsing $1 + 1 + 1 + \cdots + 1$ into k terms. This amounts to choosing a subset of $k-1$ + signs to keep from the original $n-1$. Thus,

$$\begin{aligned}P(n) &= \sum_{k=1}^{n} k\binom{n-1}{k-1} = \sum_{k=1}^{n}\binom{n-1}{k-1} + \sum_{k=1}^{n}(k-1)\binom{n-1}{k-1} \\ &= 2^{n-1} + \sum_{k=2}^{n}(n-1)\binom{n-2}{k-2} = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}.\end{aligned}$$

Thus, $P(9) = 10 \cdot 2^7 = \boxed{1280}$.

7. Let ABC be a triangle with $AB = 7$, $AC = 8$, and $BC = 9$. Let the angle bisector of A intersect BC at D. Let E be the foot of the perpendicular from C to line AD. Let M be the midpoint of BC. Find ME.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Extend CE and AB until they intersect at F. Note that AE is both an angle bisector and an altitude of $\triangle ACF$, so $\triangle ACF$ is isosceles with $AF \cong AC$, and E is the midpoint of CF. M is the midpoint of BC, so ME is a midline of $\triangle CBF$. Since $AC = AF = AB + BF$, we have $BF = 8 - 7 = 1$. Hence, $ME = \frac{1}{2}BF = \boxed{\frac{1}{2}}$.

8. Call a function g *lower-approximating* for f on the interval $[a, b]$ if for all $x \in [a, b]$, $f(x) \geq g(x)$. Find the maximum possible value of $\int_1^2 g(x)dx$ where $g(x)$ is a linear lower-approximating function for $f(x) = x^x$ on $[1, 2]$.

   Answer: $\mathbf{\frac{3\sqrt{6}}{4}}$

   Solution: We note that, because g is linear, the integral is actually the area of a trapezoid. The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2) = hM$ where M is the length of the midline. Next, $g(x) \leq f(x)$ for all $x \in [1, 2]$, and the midline has length $g(\frac{3}{2}) \leq f(\frac{3}{2}) = (3/2)^{3/2} = \frac{3\sqrt{6}}{4}$. Then $\int_1^2 g(x)dx \leq hM = \frac{3\sqrt{6}}{4}$. We note that this maximum is achieved, for example, when $g(x)$ is the tangent line to $f(x)$ at $x = \frac{3}{2}$ (which works because f is convex), so the answer must be $\boxed{\frac{3\sqrt{6}}{4}}$.

9. Determine the smallest positive integer x such that $1.24x$ is the same number as the number obtained by taking the first (leftmost) digit of x and moving it to be the last (rightmost) digit of x.

   Answer: 11415525

   Solution: Let A be an n-digit number with digits $a_1a_2 \cdots a_n$, i.e. $A = a_n + 10a_{n-1} + \cdots + 10^{n-1}a_1$. The operation of taking the first digit and moving it to the last digit results in $a_1 + 10(a_n + 10a_{n-1} + \cdots + 10^{n-2}a_2) = a_1 + 10(A - 10^{n-1}a_1)$. We need this to equal $1.24A$. Rearranging, we get

   $$A = \frac{10^n - 1}{10 - 1.24}a_1 = \frac{100(10^n - 1)}{876}a_1.$$

   $876 = 2^2 \cdot 3 \cdot 73$, and $a_1 \leq 9$, so in order for A to be an integer, we must have $73 \mid 10^n - 1 \Leftrightarrow 10^n \equiv 1 \pmod{73}$. We can compute that $10^4 \equiv -1 \pmod{73}$, so $n = 8$ is the smallest n for which $10^n \equiv 1 \pmod{73}$. Plugging in $n = 8$, we get

   $$\frac{100(10^8 - 1)}{876} = 11415525,$$

   which is in fact an integer. Hence, the smallest possible value of A occurs when $a_1 = 1$, yielding $A = \boxed{11415525}$.

10. Let a and b be real numbers chosen uniformly and independently at random from the interval $[-10, 10]$. Find the probability that the polynomial $x^5 + ax + b$ has exactly one real root (ignoring multiplicity).

    Answer: $\mathbf{\frac{45-8\sqrt[4]{2}}{45}}$

Solution: Let $f(x) = x^5 - ax + b$ (flipping the sign of a makes the analysis a bit easier without changing the answer), and consider $f'(x) = 5x^4 - a$. First, if f is monotonically increasing, it must have one real root. This case occurs if and only if $a \le 0$.

We now consider $a > 0$, and for simplicity let $\alpha = (a/5)^{1/4}$, the positive root of f'. f is increasing on $(-\infty, -\alpha)$, decreasing on $(-\alpha, \alpha)$, and increasing on (α, ∞). Hence, the only way it can have more than one root is if one such root occurs in the range $[-\alpha, \alpha]$. This will happen if and only if $0 \in [f(\alpha), f(-\alpha)]$, by the Intermediate Value Theorem.

Rewrite $f(x)$ as $(5x^4 - a)x + b - 4x^5$, so that we can easily see that $f(\alpha) = b - 4\alpha^5$, $f(-\alpha) = b + 4\alpha^5$. f has multiple roots if and only if $b - 4\alpha^5 \le 0$ and $b + 4\alpha^5 \ge 0$, i.e. $b \in [-4\alpha^5, 4\alpha^5]$. We can find the probability that b falls in this range, then take the complement to find the probability that f has exactly one root.

The area of the region satisfying this condition is twice the area under the curve $g(a) = 4(a/5)^{5/4}$ from $a = 0$ to $a = 10$ (note that $g(10) = 4 \cdot 2^{5/4} < 10$ because $(5/4)^4 = \frac{625}{256} > 2$, so this is all within the valid range for b). Thus, we find

$$2 \cdot 4 \int_0^{10} \left(\frac{a}{5}\right)^{5/4} da = \frac{8}{5^{5/4}} \cdot \frac{4}{9} 10^{9/4} = \frac{640}{9} \sqrt[4]{2}.$$

The entire feasible region for a and b is a square with area $20^2 = 400$, while the only region that yields a polynomial with multiple roots has area $\frac{640}{9}\sqrt[4]{2}$. So, report

$$1 - \frac{640\sqrt[4]{2}}{400 \cdot 9} = 1 - \frac{8\sqrt[4]{2}}{45} = \boxed{\frac{45 - 8\sqrt[4]{2}}{45}}.$$

11. Let b be a positive real number, and let a_n be the sequence of real numbers defined by $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$ for all $n > 3$. Find the smallest value of b such that

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$$

diverges.

Answer: 44

Solution: Consider the polynomial $P(x) = x^3 - x^2 - x - b$. First, we notice that this has exactly one positive root; the quickest proof is by Descartes's Rule of Signs. Let r be this root.

We now claim that there exist positive constants c_1 and c_2 such that $c_1 r^n \le a_n \le c_2 r^n$ for all $n \in \mathbb{N}$. We proceed by induction. It is easy to find such c_1 and c_2 to satisfy the base cases $n = 1, 2, 3$. Now, take $n > 3$, and assume that $c_1 r^i \le a_i \le c_2 r^i$ for all $i < n$. We see that

$$a_n = a_{n-1} + a_{n-2} + ba_{n-3} \ge c_1 r^{n-1} + c_1 r^{n-2} + c_1 b r^{n-3} = c_1 r^{n-3} \cdot (r^2 + r + b) = c_1 r^{n-3} \cdot r^3 = c_1 r^n.$$

The upper bound follows for exactly the same reason.

Finally, we examine

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}.$$

If $r < 4$, then this is bounded above by

$$\sum_{n=1}^{\infty} \frac{\sqrt{c_2} r^{n/2}}{2^n} = \sqrt{c_2} \sum_{n=1}^{\infty} \left(\frac{\sqrt{r}}{2}\right)^n,$$

which is a geometric series with common ratio $\sqrt{r}/2 < 1$, therefore convergent. However, when $r \geq 4$, the sum is bounded below by

$$\sum_{n=1}^{\infty} \frac{\sqrt{c_1} r^{n/2}}{2^n} = \sqrt{c_1} \sum_{n=1}^{\infty} \left(\frac{\sqrt{r}}{2}\right)^n \geq \sqrt{c_1} \sum_{n=1}^{\infty} 1,$$

since $\sqrt{r}/2 \geq 1$. As $c_1 > 0$, this clearly diverges.

Hence, we want to find the smallest value of b for which $r \geq 4$. Plugging in $x = 4$ to $P(x)$, we get $P(4) = 64 - 16 - 4 - b = 0 \implies \boxed{b = 44}$. It is clear that if we chose a larger value of r, the corresponding value of b would only increase, so this is the smallest value of b that makes the sum diverge.

12. Find the smallest L such that

$$\left(1 - \frac{1}{a}\right)^b \left(1 - \frac{1}{2b}\right)^c \left(1 - \frac{1}{3c}\right)^a \leq L$$

for all real numbers a, b, and c greater than 1.

Answer: $e^{-\sqrt[3]{\frac{9}{2}}}$

Solution: Let $y = \left(1 - \frac{1}{a}\right)^b \left(1 - \frac{1}{2b}\right)^c \left(1 - \frac{1}{3c}\right)^a$.

Then,

$$\log y = b \log\left(\frac{a-1}{a}\right) + c \log\left(\frac{2b-1}{2b}\right) a \log\left(\frac{3c-1}{3c}\right).$$

We know that for any x,

$$\log(x-1) - \log(x) = \int_x^{x-1} \frac{1}{t}\, dt = -\int_{x-1}^{x} \frac{1}{t}\, dt \leq -\int_{x-1}^{x} \frac{1}{x}\, dt = -\frac{1}{x}.$$

Applying this yields

$$\log y \leq -\frac{b}{a} - \frac{c}{2b} - \frac{a}{3c} \leq -\frac{3}{\sqrt[3]{6}}$$

by AM-GM. Equality holds in AM-GM if we set $b = \frac{a}{\sqrt[3]{6}}$, $c = \frac{2a}{\sqrt[3]{36}}$. We also know that

$$\begin{aligned}\lim_{x\to\infty} x(\log(x-1) - \log(x)) &= \lim_{x\to\infty} \frac{\log(x-1) - \log(x)}{\frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x-1} - \frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} -\frac{x^2}{x^2 - x} = -1.\end{aligned}$$

Hence, as a, b, and c tend to infinity in these ratios, $\log y$ will approach, but will always be bounded above by, $-\frac{3}{\sqrt[3]{6}} = -\sqrt[3]{\frac{9}{2}}$ So, the answer is $\boxed{e^{-\sqrt[3]{\frac{9}{2}}}}$.

13. Find the number of distinct ways in which $30^{\left(30^{30}\right)}$ can be written in the form $a^{\left(b^c\right)}$, where a, b, and c are integers greater than 1.

Answer: 7041

Solution: First, it is clear that $a = 30^n$ for some $n \in \mathbb{N}$. This means we have $b^c = 30^{30}/n$. In other words, $b^c = 2^i 3^j 5^k$ for integers $0 \le i, j, k \le 30$. Note that for a particular choice of i, j, and k, the possible values of c are exactly the factors of the GCD of i, j, and k (except for $c = 1$, which is not permitted by the problem statement). For each $c > 1$, c divides the GCD (i, j, k) if and only if each of i, j, and k is a multiple of c. This can happen in exactly $(\lfloor 30/c \rfloor + 1)^3 - 1$ ways, since $i = j = k = 0$ is not allowed. Hence, the answer is

$$\sum_{c=2}^{30} (\lfloor 30/c \rfloor + 1)^3 - 1 = 16^3 + 11^3 + 8^3 + 7^3 + 6^3 + 5^3 + 3 \cdot 4^3 + 5 \cdot 3^3 + 15 \cdot 2^3 - 29 = \boxed{7041}.$$

14. Convex quadrilateral $ABCD$ has sidelengths $AB = 7$, $BC = 9$, $CD = 15$. A circle with center I lies inside the quadrilateral, and is tangent to all four of its sides. Let M and N be the midpoints of AC and BD, respectively. It can be proven that I always lies on segment MN. If I is in fact the midpoint of MN, find the area of quadrilateral $ABCD$.

Answer: $11\sqrt{101}$

Solution: First, note that $DA = 13$. Let P, Q, R, S and P', Q', R', S' be the midpoints and points of tangency of AB, BC, CD, and DA respectively. Let r be the radius of circle I.

Now looking at triangle ABC, we see that QM is parallel to and half the length of AB, while looking at triangle ABD, we see that NS is also parallel to and half the length of AB. Therefore $QMNS$ is a parallelogram, so QS and MN bisect each other, which implies that they intersect at I the midpoint of MN, and $QI = IS$. As we have $IQ' = IS' = r$ and $IQ = IS$, $\triangle IQQ'$ and $\triangle ISS'$ should be congruent. If Q' and S' were on different sides of line QS, then we would have that BC and AD are parallel, but this cannot happen, as then sliding BA and CD together would result in a triangle with side lengths 7, 15 and $4(= 13 - 9)$. Thus, Q' and S' are at same side of line QS. Let $QQ' = SS' = d$, so $BQ' + AS' = (4.5 - d) + (6.5 - d) = AB = 7$. Solving for d gives $d = 2$, so the lengths of the tangents from A, B, C, and D are 4.5, 2.5, 6.5, and 8.5 respectively.

Now, let α, β, γ, and δ denote angles AIP', BIQ', CIR', and DIS', respectively. We have

$$\tan\alpha = \frac{4.5}{r}, \tan\beta = \frac{2.5}{r}, \tan\gamma = \frac{6.5}{r}, \tan\delta = \frac{8.5}{r}$$

and $\alpha + \beta + \gamma + \delta = \pi$, so by solving $\tan(\alpha + \gamma) = -\tan(\beta + \delta)$ for tangent angle sum identity we have

$$\frac{\frac{4.5}{r} + \frac{6.5}{r}}{1 - \frac{4.5 \cdot 6.5}{r^2}} = -\frac{\frac{2.5}{r} + \frac{8.5}{r}}{1 - \frac{2.5 \cdot 8.5}{r^2}},$$

$$2 = \frac{1}{r^2}(2.5 \cdot 8.5 + 4.5 \cdot 6.5)$$

so finally $r = \sqrt{101}/2$. The area formula $A = pr/2$ (p being the perimeter) gives the answer of $\boxed{11\sqrt{101}}$.

15. Marc has a bag containing 10 balls, each with a different color. He draws out two balls uniformly at random and then paints the first ball he drew to match the color of the second ball. Then he places both balls back in the bag. He repeats until all the balls are the same color. Compute the expected number of times Marc has to perform this procedure before all the balls are the same color.

Answer: 81

Solution 1: We solve the general problem where there are n balls, and claim that the answer is $(n-1)^2$.

Let's define some terms. A *path* is a fixed sequence of moves, as described in the problem statement, that terminates when all the balls are the same color. For example, one possible path might begin, "Make ball 2 color 3. Make ball 5 color 8. Make ball 8 color 1" and so on. For a path p, let $l(p)$ be the length of the path, i.e. how many moves it takes to make all balls the same color. We say that in a path, color i *wins* if, at the end of that path, all balls are of color i. Each path p also has an associated probability $P(p)$, the probability that the path will occur in this game.

Now, by definition, we are trying to compute

$$\sum_p l(p)P(p).$$

We can break this expression into n parts by rewriting it as

$$\sum_{i=1}^{n} \sum_{p\text{: color } i \text{ wins in } p} l(p)P(p).$$

By symmetry, the inner sum has the same value no matter the value of i. Hence, the problem reduces to computing this inner sum, then multiplying that by n.

Since we are conditioning on color i winning, all colors not i are indistinguishable. So, this simplifies to the same game, but with just two colors, say colors 1 and 2. The current state can be denoted by an ordered pair $(m, n-m)$, which denotes how many balls of color 1 and 2, respectively, are present in the bag. We start in state $(1, n-1)$, i.e. one ball of color 1 and $n-1$ balls of color 2, and are interested in

$$\sum_{p\text{: starts at } (1,n-1) \text{ and color 1 wins}} l(p)P(p).$$

Let

$$f(m) = \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} l(p)P(p),$$

so that we want to solve for $f(1)$. Also, let

$$g(m) = \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} P(p),$$

the probability that color 1 wins the two-color game if it starts at state $(m, n-m)$.

We now get some recurrence relations for f and g. For g, it is clear that we have $g(n) = 1$ and $g(0) = 0$. Now, for $0 < m < n$, we see that

$$g(m) = \frac{m(n-m)}{n(n-1)}g(m-1) + \frac{m(n-m)}{n(n-1)}g(m+1) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right)g(m),$$

since there's a $\frac{m(n-m)}{n(n-1)}$ probability of transitioning to $(m-1, n-m+1)$, that same probability of transitioning to $(m+1, n-m-1)$, and otherwise you stay at the same state. Rearranging, this becomes

$$2g(m) = g(m-1) + g(m+1).$$

From here, it is clear that $g(m) = \frac{m}{n}$. The quickest solution is to note that $g(1) = \frac{1}{n}$ by symmetry in the original n-color game, and to use this to compute $g(2)$, $g(3)$, etc.

Now, for f, it is clear that $f(n) = 0$ (if you start at $(n, 0)$, your path immediately halts after zero steps) and $f(0) = 0$ (because the sum is empty, as color 1 can never win). For $0 < m < n$, we have

$$\begin{aligned} f(m) &= \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} l(p)P(p) \\ &= \frac{m(n-m)}{n(n-1)} \sum_{p\text{: starts at } (m-1,n-m+1) \text{ and color 1 wins}} (l(p)+1)P(p) \\ &\quad + \frac{m(n-m)}{n(n-1)} \sum_{p\text{: starts at } (m+1,n-m-1) \text{ and color 1 wins}} (l(p)+1)P(p) \\ &\quad + \left(1 - \frac{2m(n-m)}{n(n-1)}\right) \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} (l(p)+1)P(p). \end{aligned}$$

Here, we consider taking one step from the current state. The $l(p)$ terms become $l(p) + 1$ to account for this step. Recognizing expressions for f and g, and applying our recurrence relation for g, this simplifies to

$$\begin{aligned} f(m) &= \frac{m(n-m)}{n(n-1)}(f(m-1) + g(m-1)) + \frac{m(n-m)}{n(n-1)}(f(m+1) + g(m+1)) \\ &\quad + \left(1 - \frac{2m(n-m)}{n(n-1)}\right)(f(m) + g(m)) \\ &= \frac{m(n-m)}{n(n-1)}(f(m-1) + f(m+1) + 2g(m)) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right)(f(m) + g(m)) \\ &= \frac{m(n-m)}{n(n-1)}(f(m-1) + f(m+1)) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right) f(m) + g(m) \\ &= \frac{m(n-m)}{n(n-1)}(f(m-1) + f(m+1)) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right) f(m) + \frac{m}{n}. \end{aligned}$$

Rearranging terms, we get

$$2f(m) = f(m-1) + f(m+1) + \frac{n-1}{n-m}.$$

Since our goal is to solve for $f(1)$, we start by eliminating $f(n-1)$ and work down to $f(1)$. We can do this by multiplying the above equation by the right factor, for each m. In particular, we choose

$$
\begin{aligned}
2f(n-1) &= f(n-2) + f(n) + \frac{n-1}{1} \\
2 \cdot \left(2f(n-2) \right.&= \left. f(n-3) + f(n-1) + \frac{n-1}{2}\right) \\
3 \cdot \left(2f(n-3) \right.&= \left. f(n-4) + f(n-2) + \frac{n-1}{3}\right) \\
&\cdots \\
(n-m) \cdot \left(2f(m) \right.&= \left. f(m-1) + f(m+1) + \frac{n-1}{n-m}\right) \\
&\cdots \\
(n-2) \cdot \left(2f(2) \right.&= \left. f(1) + f(3) + \frac{n-1}{n-2}\right) \\
(n-1) \cdot \left(2f(1) \right.&= \left. f(0) + f(2) + \frac{n-1}{n-1}\right).
\end{aligned}
$$

Adding these up, note that all the $f(m)$'s cancel except for $f(n)$, $f(1)$, and $f(0)$. In particular, we are left with

$$nf(1) = f(n) + (n-1)f(0) + (n-1)^2.$$

Since $f(0) = f(n) = 0$, we have $f(1) = \frac{(n-1)^2}{n}$. Plugging back into our very first equation, we get that the desired answer is $nf(1) = \boxed{(n-1)^2}$.

Solution 2: Let $\lambda_1, ..., \lambda_{41}$ be all the partitions of 10 other than 10. Notice that each coloring of the balls gives rise to a partition. Let A be the matrix whose i, j-th entry is the probability that a bag of balls in state λ_j becomes a bag of balls in state λ_i through one iteration of this process. Let x be the vector whose i-th entry is the expected number of repititions starting from state λ_i. Notice that x satisfies the recurrence $x = Ax + 1$. So all we need to do is solve the linear system $(I - A)x = 1$.

After calculating the 1681 entries of A and solving the linear system in 41 variables, we find that the answer is $\boxed{81}$ expected repititions from the initial state.