1. The coordinates of three vertices of a parallelogram are $A(1,1)$, $B(2,4)$, and $C(-5,1)$. Compute the area of the parallelogram.

   Answer: 18

   Solution: Note that the area of the parallelogram is double the area of triangle ABC. If we take AC as the base of the triangle, the height is 3, so the area is $\frac{3\cdot 6}{2} = 9$. Thus, the area of the parallelogram is $\boxed{18}$.

2. In a circle, chord AB has length 5 and chord AC has length 7. Arc AC is twice the length of arc AB, and both arcs have degree less than 180. Compute the area of the circle.

   Answer: $\frac{625\pi}{51}$

   Solution: Draw B between A and C. Because $\overset{\frown}{AC} = 2\cdot \overset{\frown}{AB}$, $BC = 5$ too. Let M be the midpoint of AC. Then

$$BM = \sqrt{5^2 - \left(\frac{7}{2}\right)^2} = \frac{\sqrt{51}}{2}.$$

   Drawing a radius to A and applying the Pythagorean Theorem,

$$r^2 = \left(\frac{7}{2}\right)^2 + \left(\frac{\sqrt{51}}{2} - r\right)^2 \implies r = \frac{25}{\sqrt{51}}$$

   so the area is $\boxed{\frac{625\pi}{51}}$.

3. Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.

   Answer: $2\sqrt{5}$

   Solution: Since the cone is symmetric, the points of tangency of the ice cream scoop to the cone make a circle on the inside of the cone. Furthermore, the center of the ice cream scoop must coincide with the center of the base of the cone. The slant height of the cone is $\sqrt{5^2 + 10^2} = 5\sqrt{5}$, so from considering similar triangles within a cross section of the cone, we get

$$\frac{5}{5\sqrt{5}} = \frac{r}{10} \implies r = \boxed{2\sqrt{5}}$$

   .

4. Let ABC be a triangle such that $AB = 3$, $BC = 4$, and $AC = 5$. Let X be a point in the triangle. Compute the minimal possible value of $AX^2 + BX^2 + CX^2$.

   Answer: $\frac{50}{3}$

   Solution: Let the perpendicular distance from X to BC and BA be x and y, respectively. Then

$$AX^2 + BX^2 + CX^2 = x^2 + y^2 + (3-x)^2 + (4-y)^2 + x^2 + y^2$$

   Completing the square gives $3\left(y - \frac{4}{3}\right)^2 + 3(x-1)^2 + \frac{50}{3}$, which has minimum $\boxed{\frac{50}{3}}$.

5. Let ABC be a triangle where $\angle BAC = 30°$. Construct D in $\triangle ABC$ such that $\angle ABD = \angle ACD = 30°$. Let the circumcircle of $\triangle ABD$ intersect AC at X. Let the circumcircle of $\triangle ACD$ intersect AB at Y. Given that $DB - DC = 10$ and $BC = 20$, find $AX \cdot AY$.

   Answer: 150

   Solution: Note that $ABDX$ and $ACDY$ are isosceles trapezoids. Thus, $AX = DB$ and $AY = DC$. Furthermore, $\angle BDC = 360° - (360° - 30° - 30° - 30°) = 90°$. Thus, $DB^2 + DC^2 = BC^2 = 400$, and $DB - DC = 10$, so $AX \cdot AY = DB \cdot DC = \dfrac{DB^2 + DC^2 - (DB - DC)^2}{2} = \boxed{150}$.

6. Let E be an ellipse with major axis length 4 and minor axis length 2. Inscribe an equilateral triangle ABC in E such that A lies on the minor axis and BC is parallel to the major axis. Compute the area of $\triangle ABC$.

   Answer: $\mathbf{\frac{192\sqrt{3}}{169}}$

   Solution: Consider a transformation that scales along the major axis by a factor of $\frac{1}{2}$ so that the ellipse becomes a circle of radius 1 and the equilateral triangle becomes an isoceles triangle. Let the transformed triangle be denoted $AB'C'$.

   Now, let $x = \frac{|B'C'|}{2}$. Then the original length $|BC| = 4x$ and since $\triangle ABC$ is equilateral, $|AB| = |BC| = 4x$. Next, we calculate the altitude dropped from A' as $1 + \sqrt{1 - x^2}$. Applying Pythagoras's theorem, we get that $|AB|^2 = (1 + \sqrt{1 - x^2})^2 + 4x^2$. Substitute $|AB| = 4x$ in and solving for x^2, we get that $x^2 = \frac{48}{169}$.

   Finally, we calculate the area of $\triangle ABC$. Since $\triangle ABC$ is equilateral, its area is given by $\frac{\sqrt{3}}{4}|BC|^2 = \frac{\sqrt{3}}{4}16x^2$. Substituting in $x^2 = \frac{48}{169}$ and simplifying, we obtain $\boxed{\dfrac{192\sqrt{3}}{169}}$.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D and E be the feet of the altitudes from A and B, respectively. Find the circumference of the circumcircle of $\triangle CDE$.

   Answer: $\mathbf{\dfrac{39\pi}{4}}$

   Solution: Let X be the intersection of AD and BE. Since $\angle CDX = \angle CEX = 90°$, $CDXE$ is a cyclic quadrilateral, and CX is the diameter of the circumcircle of CDE. Using the Law of Cosines on $\angle C$, we get

$$AB^2 = BC^2 + AC^2 - 2(AB)(BC)(\cos \angle C)$$

$$169 = 196 + 225 - 2(14)(15)(\cos \angle C)$$

$$\cos \angle C = \frac{196 + 225 - 169}{2(14)(15)} = \frac{3}{5}$$

   Therefore $\triangle CDA$ and $\triangle CEB$ are similar to a 3-4-5 triangle, and so is $\triangle XDB$. Thus we have $CD = 9$, $BD = 5$, and $DX = \frac{15}{4}$. By the Pythagorean Theorem we have $CX = \frac{39}{4}$, so the circumference of the circumcircle is

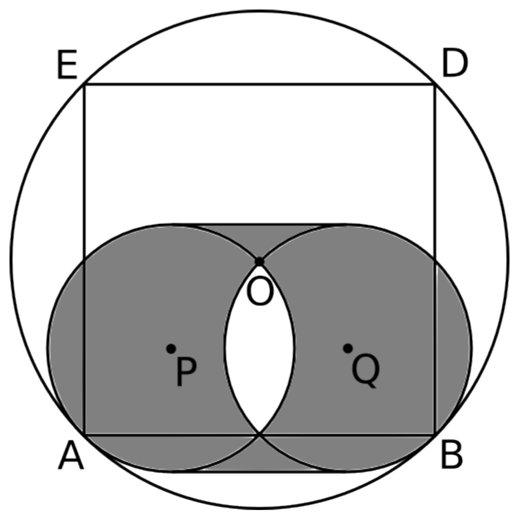
$$\pi d = \boxed{\frac{39\pi}{4}}.$$

   It is also possible to use Heron's Formula to calculate the area of the triangle and then find the lengths of the altitudes.

8. O is a circle with radius 1. A and B are fixed points on the circle such that $AB = \sqrt{2}$. Let C be any point on the circle, and let M and N be the midpoints of AC and BC, respectively. As C travels around circle O, find the area of the locus of points on MN.

   Answer: $\boldsymbol{\frac{\pi}{8} + \frac{\sqrt{2}}{2} + \frac{1}{4}}$

   Solution:



We introduce some auxilliary points which will be useful for describing the solution. It may also help to refer to the accompanying diagram. Extend AO past O to intersect the circle again at point D, and extend BO past O to intersect the circle again at point E. Note that $ABDE$ is a square of side length $\sqrt{2}$. Also let ℓ denote the perpendicular bisector of AB.

Consider what happens when C is on arc $\overset{\frown}{\mathrm{AE}}$ or $\overset{\frown}{\mathrm{BD}}$. Then, segment MN does not cross ℓ. So, the two regions corresponding to C being on these two arcs are disjoint, so we can calculate their individual areas and add them up.

As C moves from A to E (for instance), M moves from A to the midpoint of AE, and hence travels a total distance of $\frac{\sqrt{2}}{2}$ units. $MN \perp AE$ always, so this region has the same area as a rectangle with height $\frac{\sqrt{2}}{2}$ and base $\frac{1}{2}AB = \frac{\sqrt{2}}{2}$, i.e. $\frac{1}{2}$. So, the cases when C lies either on $\overset{\frown}{\mathrm{AE}}$ or $\overset{\frown}{\mathrm{BD}}$ together account for a region of area 1.

Now we consider the other cases. Let P and Q be the midpoints of OA and OB, respectively. We claim that as C moves around circle O, M traces out a circle of radius $\frac{1}{2}$ centered at P, and N traces out a circle of radius $\frac{1}{2}$ centered at Q. As proof, note that MP is the midline of triangle AOC that is parallel to OC. Since $OC = 1$, $MP = \frac{1}{2}$ always (alternatively, note that we have a homothety from CD to MP with ratio $\frac{1}{2}$).

So, when C lies on $\overset{\frown}{\mathrm{DE}}$, we can see that the region covered by MN consists of a circle segment of angle $\frac{\pi}{2}$ broken in the middle by a rectangle of width $\frac{\sqrt{2}}{2}$. The height of this rectangle can be quickly computed to be $\frac{1}{2} - \frac{\sqrt{2}}{4}$. Hence, the total area here is

$$\frac{1}{4} \cdot \pi \cdot \left(\frac{1}{2}\right)^2 - \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) \cdot \frac{\sqrt{2}}{2} = \frac{\pi}{16} + \frac{\sqrt{2}}{4} - \frac{3}{8}.$$

The case where C lies on $\overset{\frown}{\text{AB}}$ turns out to be equivalent. Hence, in total, the desired area is

$$1 + 2\left(\frac{\pi}{16} + \frac{\sqrt{2}}{4} - \frac{3}{8}\right) = \boxed{\frac{\pi}{8} + \frac{\sqrt{2}}{2} + \frac{1}{4}}.$$

9. In cyclic quadrilateral $ABCD$, $AB \cong AD$. If $AC = 6$ and $\frac{AB}{BD} = \frac{3}{5}$, find the maximum possible area of $ABCD$.

   Answer: $\mathbf{5\sqrt{11}}$

   Solution: We claim that the area of $ABCD$ is constant, though the shape of $ABCD$ depends on the radius of the circumscribing circle.

   First, by Ptolemy's, we have

$$AB \times CD + BC \times AD = AB(BC + CD) = AC \times BD$$
$$\implies BC + CD = AC \times \frac{BD}{AB} = 10.$$

   Now, extend CB past B to a point E such that $BE \cong CD$. Since $ABCD$ is cyclic, $\angle ABC$ and $\angle ADC$ are supplementary, so $\angle ADC \cong \angle ABE$. Hence, $\triangle ADC \cong \triangle ABE$, and so $ABCD$ has the same area as ACE.

   ACE is isosceles with leg 6 and base $CE = CB + BE = CB + DC = 10$. Hence, the length of the altitude to the base is $\sqrt{6^2 - 5^2} = \sqrt{11}$, and the area is $\boxed{5\sqrt{11}}$.

10. Let ABC be a triangle with $AB = 12$, $BC = 5$, $AC = 13$. Let D and E be the feet of the internal and external angle bisectors from B, respectively. (The external angle bisector from B bisects the angle between BC and the extension of AB.) Let ω be the circumcircle of $\triangle BDE$; extend AB so that it intersects ω again at F. Extend FC to meet ω again at X, and extend AX to meet ω again at G. Find FG.

    Answer: $\mathbf{\frac{1560}{119}}$

    Solution: Note that $\angle DBE = 90°$, so DE is the diameter of ω. Let BC intersect ω at G'. Since $\overset{\frown}{\text{G'E}} = \overset{\frown}{\text{FE}} = 90°$, $G'XBF$ is an isosceles trapezoid by symmetry over the diameter DE. By the same symmetry, $\triangle AXC \cong \triangle ABC$, so $\angle AXC = \angle ABC = 180° - \angle G'BF = 180° - \angle CXG'$; thus A, X, and G' are collinear, so $G = G'$. Note that $\angle GBF = 180° - \angle ABC = 90°$, so FG is a diameter; thus $FG = DE$. Now we calculate DE. Since BE is an external angle bisector,

$$\frac{CE}{AE} = \frac{BC}{AB}$$
$$\frac{CE}{AC + CE} = \frac{BC}{AB}$$
$$CE = \frac{AC \cdot BC}{AB - BC} = \frac{65}{7}.$$

    Since BD is an internal angle bisector,

$$\frac{CD}{AD} = \frac{BC}{AB}$$
$$\frac{CD}{AC} = \frac{BC}{AB + BC}$$
$$CD = \frac{AC \cdot BC}{AB + BC} = \frac{65}{17}.$$

Thus, the answer is $DE = CE + CD = \boxed{\frac{1560}{119}}$.