1. **Answer: 4**

   Solution: We know that $2013^{2013} = 3^{2013} \cdot 11^{2013} \cdot 61^{2013}$. Therefore, $f_1(2013^{2013}) = 2014^3$. $2014 = 2 \cdot 19 \cdot 53$, so 2014^3 has $4^3 = 64$ divisors. $f_1(64) = 7$, and $f_1(7) = 2$. This means that $f_4(2013^{2013}) = 2$, so $k = \boxed{4}$.

2. **Answer: $\mathbf{\frac{1}{12}}$**

   Solution: Let (ABC) denote the area of polygon ABC. Note that $\triangle AFB \sim \triangle MFD$ with $AB/MD = 2$, so we have $DF = \frac{1}{3}BD$. This implies that $(MFD) = \frac{1}{3}(MBD) = \frac{1}{3}(\frac{1}{2}(CBD)) = \frac{1}{12}$. By symmetry, $(MGC) = \frac{1}{12}$ as well. Therefore, we have $(MFEG) = (CED) - (MBD) - (MGC) = \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \boxed{\dfrac{1}{12}}$.

3. **Answer: 280**

   Solution 1: Given a permutation of nine people, let us have the first three people be in one group, the second three people in another group, and the last three people in a third group. We want to compute how many permutations generate the same group. Note that there are $(3!)^3$ ways to permute people within each group, and there are 3! ways to permute the overall groups, so the answer is $\dfrac{9!}{(3!)^4} = \boxed{280}$.

   Solution 2: Note that if three people are doing this, there is trivially exactly one unique iteration.

   If six people are doing this, then arbitrarily label one person. There are $\binom{5}{2}$ groups that can be created with this person, and then the other three people are forced to be in a group, so there are $\binom{5}{2}$ iterations for six people.

   If nine people are doing this, then arbitrarily label one person. There are $\binom{8}{2}$ groups that can be created with this person, and then the other six people can form groups in $\binom{5}{2}$ ways, so there are $\binom{5}{2} \cdot \binom{8}{2} = \boxed{280}$ iterations for nine people.

4. **Answer: $\mathbf{(7, 3)}$**

   Solution: The first term is just $(x^a)^b(x^3)^a = x^{ab+3a}$, so $ab + 3a = 42$. Using the binomial theorem, the second term is $(\binom{b}{1}(x^a)^{b-1}(abx^{a-1}))(x^3)^a + (\binom{a}{1}(x^3)^{a-1}(3bx^2))(x^a)^b = (ab^2 + 3ab)x^{ab+3a-1}$, so $ab^2 + 3ab = 126$. Factoring these two equations gives $a(b + 3) = 42$ and $ab(b+3) = 126$. Dividing the second equation by the first gives $b = \boxed{3}$. Then, substituting that into the first equation gives $a = \boxed{7}$.

5. **Answer: $\boldsymbol{(n + 2)2^{n-1}}$**

   Solution: There are 2^n cases, which can be considered based on their divisibility by powers of 2. Suppose we twist the prism by $\frac{x}{2^n}$ of a full rotation where $x = 2^k \cdot y$ and y is odd. Note that under this twist, each of the original 2^n faces is linked to every x^{th} face. Since y is odd, we see that the first multiple of x divisible by 2^n is $2^{n-k} \cdot x$. Thus, each face in the new figure is made up of 2^{n-k} faces from the original (untwisted) form and there are $\frac{2^n}{2^{n-k}} = 2^k$ sides for this figure. Next, we must consider how many rotations have x of the form $2^k \cdot y$ for a fixed value of k. The number of integers less than or equal to 2^n that are divisible by 2^k and not 2^{k+1} is $\frac{2^n}{2^k} - \frac{2^n}{2^{k+1}}$. For $k < n$, this equals 2^{n-k-1}. Finally, we add in the untwisted case, with 2^n faces. Thus, the total number of sides is $\sum_{k=0}^{n-1} 2^k \cdot 2^{n-k-1} + 2^n = n2^{n-1} + 2^n = \boxed{(n + 2)2^{n-1}}$.

6. **Answer: 4**

   Solution: Let $a_1, \ldots, a_5$ denote the 5 elements of A in increasing order. Let S denote $\sum_{i=1}^{5} a_i$. First, note that $1, 2 \in A$ because there are no other ways to obtain 1 and 2. Hence, we only have to think about the other three elements of A.

   Note that there are $2^5 - 1 = 31$ non-empty subsets of A, so at most two of those subsets are not useful to the subset-sum constraint, either by having sum greater than 29 or being redundant with another subset.

   We condition on the value of S. This sum is clearly at least 29, and must be at most 31, since S, $S-1$, and $S-2$ are all achievable subset-sums, so we require $S - 2 \leq 29$.

   If $S = 31$, then $A \setminus \{1\}$ has sum 30, so each subset has a distinct sum. Therefore, $4 \in A$ because the only other way to get 4 would be $1 + 3$, but $3 \in A$ would imply there were two different ways to get 3, namely $1 + 2$ and 3. Similarly, since the subsets of $\{1, 2, 4\}$ can sum to any positive integer less than 8, $8 \in A$. $16 \in A$ for the same reason for the set $\{1, 2, 4, 8\}$, and so A is completely determined.

   If $S = 30$, we may have exactly one pair of subsets with the same sum. Hence, we still get $4 \in A$ because $3 \in A$ would imply $a_1 + a_2 = a_3$ and $a_1 + a_2 + a_5 = a_3 + a_5$. Similarly, $8 \in A$. Finally, since we know all elements of A must sum to 30, we choose $a_5 = 15$.

   If $S = 29$, then we still have $4 \in A$ because there will be more than two redundant pairs of subsets if $3 \in S$. In general, we cannot have $x + y = z$ for $x, y, z \in A$ because there would be too many redundant sets. Hence, $a_4 \geq 7$. It can be at most 8, since otherwise there would be no way to achieve a sum of 8, so there are two cases for a_4. Each choice of a_4 determines a_5 by the condition on S. We can verify that $a_4 = 7, a_5 = 15$ works because $\{1, 2, 4\}$ can generate all sums ≤ 7, so $\{1, 2, 4, 7\}$ can generate all sums ≤ 14. Adding 15 clearly yields all sums ≤ 29. The other case can be checked trivially.

   Hence, in total there are $\boxed{4}$ viable sets.

7. **Answer:** $\mathbf{(-1, 0) \cup \frac{1}{4}}$

   Solution: There are two possibilities: either the curves $y = x^2 + u$ and $x = y^2 + u$ intersect in exactly one point, or they intersect in two points but one of the points occurs on the branch $y = -\sqrt{x - u}$.

   Case 1: the two curves are symmetric about $y = x$, so they must touch that line at exactly one point and not cross it. Therefore, $x = x^2 + u$, so $x^2 - x + u = 0$. This has exactly one solution if the discriminant, $(-1)^2 + 4(1)(u) = 1 + 4u$, equals 0, so $u = \boxed{\frac{1}{4}}$.

   Case 2: $y = x^2 + u$ intersects the x-axis at $\pm\sqrt{-u}$, while $y = \sqrt{x - u}$ starts at $x = u$ and goes up from there. In order for these to intersect in exactly one point, we must have $-\sqrt{-u} < u$, or $-u > u^2$ (note that $-u$ must be positive in order for any intersection points of $y = x^2 + u$ and $x = y^2 + u$ to occur outside the first quadrant). Hence we have $u(u + 1) < 0$, or $u \in (-1, 0)$.

8. **Answer:** $\mathbf{\frac{\sqrt{33}-3}{3}}$

   Solution: We can un-parametrize this equations easily to see that Rational Man is traveling along the circle
$$x^2 + y^2 = 1$$

with a period of 2π, while Irrational Man is travelling along the ellipse

$$\frac{(x-1)^2}{16} + \frac{y^2}{4} = 1$$

with a period of $2\pi\sqrt{2}$.

Now, we claim that d is equal to the smallest distance between a point on the given circle and a point on the given ellipse. This is because for any number $r \in [0, 1)$, we can find a positive integer multiple of $\sqrt{2}$ whose fractional part is arbitrarily close to r, using a Pigeonhole argument. More precisely, for any $n \in \mathbb{N}$, we consider $\sqrt{2}, 2\sqrt{2}, \ldots, n\sqrt{2}$. Now divide the region between 0 and 1 into n equally-spaced intervals. For a given $r \in [0, 1)$, find the interval it falls into. Either one of our n multiples of $\sqrt{2}$ falls into this interval (and thus is at most $\frac{1}{n}$ from r), or none of them do, in which case two numbers fall into the same interval, and thus their difference has fractional part of magnitude less than $\frac{1}{n}$. Now it is clear that we can take a multiple of this number that is within $\frac{1}{n}$ of r.

Now consider Rational Man's position at any time t. This is the same as his position at time $t + 2\pi n$ for all $n \in \mathbb{N}$. Now, if Irrational Man assumes some position at time t', then he also assumes it at time $t' + 2\pi m\sqrt{2}$ for all $m \in \mathbb{N}$. By the fact proven above, we can always choose an m such that $t' + 2\pi m\sqrt{2}$ is arbitrarily close to $t + 2\pi n$ for some $n \in \mathbb{N}$ (divide through by 2π to make this clearer). Since the two drivers can get arbitrarily close to any pair of points on their respective paths, d must simply be the shortest distance between these two paths.

Now we make the observation that given a circle of radius r centered at O and a point P outside this circle, the shortest distance from P to the circle is along the line that passes through O. This is evident by applying the Triangle Inequality to triangle OPQ, where Q is any point on the circle that is not on the line OP. Hence, minimizing distance between the ellipse and the circle is equivalent to minimizing distance between the ellipse and the center of the circle, i.e. the origin.

Hence, we set out to minimize $x^2 + y^2$ subject to the constraint $\frac{(x-1)^2}{16} + \frac{y^2}{4} = 1$. Thus, we are minimizing

$$x^2 + 4 - \frac{1}{4}(x-1)^2 = \frac{1}{4}(3x^2 + 2x + 15).$$

This attains its minimum value at $x = -\frac{1}{3}$, so the minimum squared distance from the origin is $\frac{11}{3}$. We want one less than the distance to the origin as our final answer, so report $\boxed{\dfrac{\sqrt{33}-3}{3}}$.

9. **Answer:** $\mathbf{\frac{14951}{150}}$

Solution: We claim that when 100 is replaced by n, the answer is $n - \frac{n-2}{3n} = \frac{3n^2-n+2}{3n}$.

By symmetry and linearity of expectation, we need only compute the expected value of y, then multiply by two.

First, each $i = 1, \ldots, n-1$ is seen $n-2$ times (once in each row except for row i), while n is present in every row. Hence, the sum of all values is

$$n(n-1) + (n-2)\sum_{i=1}^{n-1} i = n(n-1) + \frac{1}{2}n(n-1)(n-2) = \frac{1}{2}n^2(n-1).$$

Meanwhile, the sum of values in row i has weight

$$\frac{1}{2}n(n+1) - i = \frac{n(n+1)-2i}{2}.$$

Hence, the desired expectation is

$$\sum_{i=1}^{n-1} \frac{(n(n+1)-2i)}{n^2(n-1)} i = \frac{n+1}{n(n-1)} \sum_{i=1}^{n-1} i - \frac{2}{n^2(n-1)} \sum_{i=1}^{n-1} i^2 = \frac{n+1}{2} - \frac{2n-1}{3n} = \frac{3n^2-n+2}{6n}.$$

Multiplying by two gives the final answer.

10. **Answer:** $\boldsymbol{\frac{\pi(2+\sqrt{3})^2}{8\sqrt{3}} = \frac{\pi(7+4\sqrt{3})}{8\sqrt{3}} = \pi\left(\frac{1}{2}+\frac{7}{8\sqrt{3}}\right) = \pi\left(\frac{1}{2}+\frac{7\sqrt{3}}{24}\right)}$

Solution: Let $\alpha = 5\pi/6$. First, notice that because the tangent line has constant slope, the intersection point on every circle must occur at the same angle with respect to the circle's center. Likewise, the line between the intersection points on two circles must coincide with the tangent line. Let circle 1 have radius R and center at $(X, 0)$ and circle 2 have radius r. It follows that circle 2 has a center at $(X+R+r, 0)$. Thus the two points of intersection with the tangent line are $(X+R\cos\alpha, R\sin\alpha)$ and $(X+R+r+r\cos\alpha, r\sin\alpha)$. The line between these must have slope $-\cot\alpha$, so

$$\frac{r\sin\alpha - R\sin\alpha}{X+R+r+r\cos\alpha-(X+R\cos\alpha)} = -\cot\alpha \implies r = R\tan^2\alpha/2.$$

Clearly, $\alpha < \pi/2$ so $\tan^2\alpha/2 < 1$. Thus the radii obey a geometric series:

$$\sum_{n=0}^{\infty} \pi[\tan^{2n}\alpha/2]^2 = \cos^4\left(\frac{\alpha}{2}\right)\sec\alpha.$$

Plugging in $\alpha = 5\pi/6$ gives $\boxed{\frac{\pi(2+\sqrt{3})^2}{8\sqrt{3}}}$.

11. **Answer:** $\mathbf{73513440 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}$

Solution: Note that $768 = 2^8 \cdot 3$. We can immediately upper bound the answer to $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. It may be possible to increase exponents on small primes and discard larger primes to reduce the answer.

There are a few cases to consider.

(a) 5 is the largest power of 5 that divides the answer. Therefore, one of 2 and 3 must contribute the factor of 3 to the number of divisors. We have two subcases to consider at this point:

   i. 2 contributes the factor of 3. We initially set $2^2 \cdot 3$. We can destroy 19 and 23 by using 2^5 and 3^3.

   ii. 3 contributes the factor of 3. We must use 3^2, and therefore the power of 2 should be 2^7, destroying 19 and 23 also.

(b) 25 is the largest power of 5 that divides the answer. We must therefore use at least 2^3 and 3^3.

Note that the very first subcase generates the smallest product, so the answer is therefore $\boxed{2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}$.

12. **Answer:** $\boldsymbol{t = \frac{3}{2\pi} \arccos\left(\frac{r+\sqrt{r^2+8R^2}}{4R}\right)}$.

Solution: Define $\omega_1 = \frac{2\pi}{3}$ and $\omega_2 = 2\pi$. Let $(x_1(t), y_1(t))$ be the location of Robin and $(x_2(t), y_2(t))$ be the location of Eddy at time t. Let the center of the path be the origin and Jack's location be $(R, 0)$. Then we have

$$x_1 = r\cos(\omega_1 t) \qquad y_1 = r\sin(\omega_1 t)$$
$$x_2 = r\cos(\omega_2 t) \qquad y_2 = r\sin(\omega_2 t).$$

The three people are collinear if and only if the slopes of the lines connecting any two people are the same, i.e.

$$\frac{y_1}{x_1 - R} = \frac{y_2}{x_2 - R}.$$

Plugging in and simplifying gives us $r\sin(\omega_2 t - \omega_1 t) = R\sin(\omega_2 t) - R\sin(\omega_1 t)$. Noting that $3\omega_1 = \omega_2$ and using sum-to product identites, we get

$$r\sin(\omega_1 t)\cos(\omega_1 t) = R\sin(\omega_1 t)\cos(2\omega_1 t).$$

So either the first time the three people are collinear is when $t = \pi$ or when $r\cos(\omega_1 t) = R\cos(2\omega_1 t)$. Using the double angle identity for cosine gives us a quadratic in cosine. We can apply the quadratic formula to get

$$\cos(\omega_1 t) = \frac{r \pm \sqrt{r^2 + 8R^2}}{4R}.$$

The positive root is the only collinear time that occurs when Robin is still in the first quadrant. Therefore, it is the earliest time. $r < R$ implies

$$\cos(\omega_1 t) = \frac{r + \sqrt{r^2 + 8R^2}}{4R} < 1,$$

so it is in the range of the cosine function. Hence, the answer is

$$\boxed{t = \frac{3}{2\pi} \arccos\left(\frac{r + \sqrt{r^2 + 8R^2}}{4R}\right)}.$$

13. **Answer: 22**

Solution: We claim that N can be written on the board if and only if $N + 1$ has a prime factorization of the form $3^a 5^b 7^c$, where $a + b + c$ is odd. It remains to actually prove this.

Note that if we write $N = xyz + xy + yz + zx + x + y + z$, then we have that $N + 1 = (x + 1)(y + 1)(z + 1)$. Note that the original numbers, 2, 4, and 6, are each less than the primes 3, 5, and 7, respectively. Therefore, we ensure that the only primes which can divide any valid $N + 1$ are 3, 5, and 7. Furthermore, these numbers each have exponents summing to 1, an odd integer, so therefore since we multiply three integers with an odd sum of exponents, we ensure that all numbers which remain have an odd sum of exponents.

It remains to compute all numbers of the form $3^a 5^b 7^c$, where each number is less than or equal to 2013 and the sum of the exponents is odd. There are $\boxed{22}$ such numbers.

14. **Answer: $8 - 4 \log 2$**

Solution: Let $f(x)$ be the average number of cuts you make if you start with x meters of string. For $x \in [0, \frac{1}{2})$, we have $f(x) = 0$.

To calculate $f(x)$ for $x \geq \frac{1}{2}$, say you make 1 cut that brings the length of the string to y. Then you average a total of $1 + f(y)$ cuts. Note that y is distributed uniformly at random from $\frac{x}{2}$ to x. So we average $1 + f(y)$ over $y \in [\frac{x}{2}, x]$, which gives us

$$f(x) = \frac{2}{x} \int_{x/2}^{x} f(y)\, dy + 1.$$

Now we have an initial condition and recurrence. To solve this, let $F(x) = \int_0^x f(t)\, dt$. In terms of F, the recurrence is

$$F'(x) = \frac{2}{x} \left(F(x) - F(x/2)\right) + 1.$$

For $x \in [\frac{1}{2}, 1)$, the initial condition tells us that $F(x/2) = 0$, so the recurrence simplifies to

$$F'(x) = \frac{2}{x} F(x) + 1.$$

Also notice that we have an initial condition $F(\frac{1}{2}) = 0$. Since multiplying by $\frac{2}{x}$ is the same as differentiation for x^2, we might guess that a degree 2 polynomial solves this differential equation. If we plug in a general degree 2 polynomial, we find that we are correct and that the solution is $F(x) = 2x^2 - x$ for $x \in [\frac{1}{2}, 1)$. Plugging this into our original recurrence for F, we get the following differential equation, now valid for $x \in [1, 2]$:

$$F'(x) = \frac{2}{x} F(x) - x + 2.$$

This time we could try another degree 2 polynomial, but it won't work. We need to somehow get a term involving x on one side without getting it on the other side in order to balance the x's on each side. Notice that $x^2 \log x$ will give us an x when we differentiate but not when we multiply by $\frac{2}{x}$. So that might work. And indeed it does. We can plug in $F(x) = ax^2 + bx + cx^2 \log x$, solve for the coefficients (keeping in mind the initial condition $F(1) = 1$ that we get from our previous expression for F), and get $F(x) = 3x^2 - x^2 \log x - 2x$.

Now we know $F(x)$ on all of $[0, 2]$, so we can differentiate it to get $f(x)$. The result is

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 4x - 1, & \text{if } x \in [\frac{1}{2}, 1) \\ 5x - 2x \log x - 2, & \text{if } x \in [1, 2] \end{cases}$$

Therefore the answer is $f(2) = 8 - 4 \log 2$.

15. **Answer: $\sqrt{2504 \log^2 2 + 2500\pi^2}$**

Solution: For ease of notation, let $r_0 = 100$ and $h = 4$. Begin by flattening the cone into a sector of a circle with radius $R = \sqrt{r_0^2 + h^2}$. The problem then is equivalent to finding the optimal path from the polar point $(r, \theta) = (R, 0)$ to the point $(\frac{R}{2}, \frac{r_0}{R} \cdot \pi)$ on the flattened cone. We can find an optimal path by constructing a new "distance" metric that measures elapsed time by considering standard Euclidean distance along with a factor that accounts for velocity.

Observe that any point on the sector with "radius" (distance along the cone's surface to the center) r and height (on the cone) z satisfies $\frac{h-z}{h} = \frac{r}{R}$ by similar triangles. Therefore, the speed at radius r on the sector is $\frac{r}{R} \cdot v_0$.

Let the optimal path curve be given by $\gamma(\theta) = (r(\theta), \theta)$. We wish to optimize the integral that gives the total time spent along the curve γ. We can measure length by the standard polar arclength formula, and we can measure speed using the formula above. Hence, we can measure time by looking at distance divided by speed:

$$\int_{\theta=0}^{\frac{r_0}{R}\cdot\pi} \frac{\text{distance}}{\text{speed}} = \int_0^{\frac{r_0}{R}\cdot\pi} \frac{\sqrt{dr^2 + r^2\, d\theta^2}}{\frac{r}{R}\cdot v_0} = \frac{R}{v_0}\int_0^{\frac{r_0}{R}\cdot\pi} \sqrt{1 + \left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2}\, d\theta.$$

We now wish to find a coordinate transformation in which this path is a straight line, so that the minimum time will just be the Euclidean distance between the endpoints. We can do this by choosing a new coordinate $\tilde{r}$ so that $\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{d\tilde{r}}{d\theta}$. By integrating, it is easily seen that one such substitution is $\log r = \tilde{r}$, which results in the endpoints

$$(\log R, 0) \text{ and } \left(\log\frac{R}{2}, \frac{r_0}{R}\cdot\pi\right),$$

so our integral is

$$\sqrt{\left(\log R - \log\frac{R}{2}\right)^2 + \frac{r_0^2}{R^2}\cdot\pi^2} = \sqrt{(\log 2)^2 + \frac{r_0^2}{R^2}\cdot\pi^2}.$$

Multiplying by the constant terms we factored out of this integral earlier, our final answer (and minimum time) is

$$\sqrt{\frac{R^2}{v_0}(\log 2)^2 + \frac{r_0^2}{v_0^2}\cdot\pi^2} = \boxed{\sqrt{2504(\log 2)^2 + 2500\pi^2}}.$$