1. (a) (i) $\{0, 5, 7, 10, 11, 12, 14\} \cup \{n \in \mathbb{N}_0 : n \geq 15\}$.

   (ii) Yes, $\langle 5, 7, 11, 16\rangle$ can be generated by a set of fewer than 4 elements. Specifically, it is generated by $\{5, 7, 11\}$ because $16 = 11 + 5$ and therefore any 16's in an element of the semigroup can be written using 5's and 11's.

   (iii) $\{0, 3, 6, 7, 8\} \cup \{n \in \mathbb{N}_0 : n \geq 9\}$.

   (iv) No, $\langle 3, 7, 8\rangle$ cannot be generated by a set of fewer than 3 elements. If this were possible, then we could write $\langle 3, 7, 8\rangle = \langle a, b\rangle$ for two integers $a < b$. For this to work, we must have $a = 3$. (If $a < 3$, then $\langle a, b\rangle$ contains $a \notin \langle 3, 7, 8\rangle$. If $a > 3$, then $\langle a, b\rangle$ doesn't contain anything that can generate a 3. The only possibility left is $a = 3$.) Furthermore, we must have $b = 7$. (If $b < 6$, then $\langle 3, b\rangle$ contains $b \notin \langle 3, 7, 8\rangle$. If $b > 7$, then $\langle 3, b\rangle$ doesn't contain anything that can generate a 7. And $b = 6$ is not allowed because $\gcd(3, 6) > 1$. The only possibility left is $b = 7$.) So if we can generate $\langle 3, 7, 8\rangle$ using fewer than 3 elements, then $\langle 3, 7, 8\rangle = \langle 3, 7\rangle$. This is not true, because $8 \notin \langle 3, 7\rangle$. Therefore the answer is no, as we claimed.

   (b) Suppose $x, y \in \langle a_1, \ldots, a_n\rangle$. Then there are $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{N}_0$ such that $x = c_1a_1 + \cdots + c_na_n$ and $y = d_1a_1 + \cdots + d_na_n$. Then $x + y = (c_1 + d_1)a_1 + \cdots + (c_n + d_n)a_n$, which is in $\langle a_1, \ldots, a_n\rangle$ by definition.

   (c) Let $d_1, \ldots, d_n$ be integers such that $d_1a_1 + \cdots + d_na_n = 1$. Let $M = \max |d_i|$. Let $s = a_1Ma_1 + a_1Ma_2 \cdots + a_1Ma_n$. Then for any $0 \leq r < a_1$, all the coefficients in $s + r = (a_1M + rd_1)a_1 + (a_1M + rd_2)a_2 + \cdots + (a_1M + rd_n)a_n$ are positive and therefore $s + r \in \langle a_1, \ldots, a_n\rangle$. Any integer $x \geq s$ can be written as $x = qa_1 + (s + r)$ with $r < a_1$ by letting q be the quotient of $\frac{x-s}{a_1}$ and by letting r be the remainder. Now $qa_1 \in \langle a_1, \ldots, a_n\rangle$ by definition and we have shown $(s + r) \in \langle a_1, \ldots, a_n\rangle$, so the sum $x = qa_1 + (s + r)$ is also in $\langle a_1, \ldots, a_n\rangle$. Since every integer $x \geq s$ is in $\langle a_1, \ldots, a_n\rangle$, there are only finitely many positive integers not in $\langle a_1, \ldots, a_n\rangle$.

2. (a) We will try to keep adding smallest un-generated elements to our set of generators until we get a set of generators that generate everything. To do this, let $A_0 = \emptyset$. If $S - \langle A_i\rangle$ is nonempty, then let A_{i+1} be A_i unioned with the smallest element of $S - \langle A_i\rangle$. (Where $\langle A\rangle$ denotes the set of all $\mathbb{N}_0$-linear combinations of elements of A). If $S - \langle A_i\rangle$ is empty, then let $A_{i+1} = A_i$.

   First we claim that every integer in S that is less than i is in $\langle A_i\rangle$. We can show this by induction. For the base case, every integer in S less than 0 is in $\langle A_0\rangle$. For the inductive step, suppose every integer in S that is less than i is in $\langle A_i\rangle$. We would like to show that every integer in S that is less than $i + 1$ is in $\langle A_{i+1}\rangle$. Since $\langle A_i\rangle \subseteq \langle A_{i+1}\rangle$, every integer in S that is less than i is already in $\langle A_{i+1}\rangle$. So we only need to show that if $i \in S$ then $i \in \langle A_{i+1}\rangle$. If $i \in S$ and $i \in \langle A_i\rangle$, then $i \in \langle A_{i+1}\rangle$ and we are done. If $i \in S$ and $i \notin \langle A_i\rangle$, then i is the smallest number in $S - \langle A_i\rangle$ and therefore $i \in \langle A_{i+1}\rangle$ by construction. So we have proven the claim.

   Let's show that there is an n such that $S = \langle A_n\rangle$. To do this, let $p < q$ be two distinct primes in S (there are at least two distinct primes in S because all but finitely many positive integers are in S). By our claim, $p, q \in A_{q+1}$. Since $\gcd(p, q) = 1$, the set $\langle p, q\rangle$ contains all but finitely many positive integers (we proved this in 1c). Since $\langle p, q\rangle \subset \langle A_{q+1}\rangle$, this means that $\langle A_{q+1}\rangle$ contains all but finitely many positive integers. In particular, $S - \langle A_{q+1}\rangle$ is finite. So there is an integer $n > q + 1$ that is bigger than all the elements in $S - \langle A_{q+1}\rangle$. By our above claim, $\langle A_n\rangle$ contains all of $S - \langle A_{q+1}\rangle$. And since $n > q+1$, $\langle A_n \rangle$ also contains all of $\langle A_{q+1} \rangle$. Therefore $\langle A_n \rangle \supseteq S$. By construction, $\langle A_n \rangle \subseteq S$. Therefore we have equality $\langle A_n \rangle = S$ as desired.

So now we have a set of integers $A_n = a_1, \ldots, a_n$ with $\langle a_1, \ldots, a_n \rangle = S$. We have almost shown what we wanted. But we must still show that $\gcd(a_1, \ldots, a_n) = 1$. To do this, assume for a contradiction that $\gcd(a_1, \ldots, a_n) > 1$. Let $d = \gcd(a_1, \ldots, a_n)$. Then $d > 1$ divides everything in $\langle a_1, \ldots, a_n \rangle$, and so there are infinitely many positive integers not in $\langle a_1, \ldots, a_n \rangle$. This is a contradiction and therefore $\gcd(a_1, \ldots, a_n) = 1$.

(b) The A_n constructed above is the unique minimal generating set. To see this, let $a_1 < a_2 < \ldots < a_N$ be the elements of A_n and let $b_1 < b_2 < \ldots < b_m$ be the elements of a minimal generating set. We will prove that the sequences a_i, b_i are equal. First notice that $N \geq m$ because $b_1, \ldots, b_m$ is a minimal generating set. We will therefore start by showing that $a_i = b_i$ for all $i \leq m$.

Assume for a contradiction that there is some $i \leq m$ such that $a_i \neq b_i$. Let i be the minimum such i. Since $\{b_1, ..., b_m\}$ is minimal, $b_i \notin \langle b_1, \ldots, b_{i-1} \rangle$. In other words, $b_i \in S - \langle b_1, \ldots, b_{i-1} \rangle$.

Furthermore, we claim that b_i is the smallest element in $S - \langle b_1, \ldots, b_{i-1} \rangle$. To see this, let r be the smallest element in $S - \langle b_1, \ldots, b_{i-1} \rangle$. Then r is some nonnegative linear combination of $b_1, \ldots, b_n$ involving at least one element past b_{i-1}. If $b_i > r$, then all elements past b_{i-1} are greater than r and therefore r cannot be made with such a nonnegative linear combination. Therefore $b_i \leq r$, forcing $b_i = r$ as desired. b_i is the smallest element in $S - \langle b_1, \ldots, b_{i-1} \rangle$.

Since we chose i to be the minimum such that $a_i \neq b_i$, we know that $\langle a_1, \ldots, a_{i-1} \rangle = \langle b_1, \ldots, b_{i-1} \rangle$. In particular, $S - \langle b_1, \ldots, b_{i-1} \rangle = S - \langle a_1, \ldots, a_{i-1} \rangle$. So b_i is the smallest element in $S - \langle a_1, \ldots, a_{i-1} \rangle$. But this is exactly how we defined a_i! Therefore $a_i = b_i$, contradicting our assumption that $a_i \neq b_i$.

So we have proven by contradiction that $a_i = b_i$ for all $i \leq m$. Since $\{b_1, \ldots, b_m\}$ generates S, our construction of A_n stops adding elements once it gets to a_m. So the sequence $a_1, \ldots, a_N$ actually has m elements and we are done.

3. (a) (i) Genus 8, Frobenius 13. (ii) Genus 4, Frobenius 5.

(b) Let N be any positive integer that is not in the semigroup. Then at least one number from each of the the $\lfloor \frac{N}{2} \rfloor$ pairs $(1, N-1), (2, N-2), \ldots, (\lfloor \frac{N}{2} \rfloor, \lceil \frac{N}{2} \rceil)$ must not be in the semigroup. Also, N is not in the semigroup. So there are at least $\lfloor \frac{N}{2} \rfloor + 1 \geq \frac{N+1}{2}$ positive integers not in the semigroup. Ie, $g(S) \geq \frac{N+1}{2}$. Plugging in $N = F(S)$, we get $g(S) \geq \frac{F(S)+1}{2}$. Rearranging, $F(S) \leq 2g(S) - 1$.

4. (a) $F(\langle a, b \rangle) = ab - a - b$.

(b) We will use the following fact: If (x_0, y_0) is an integer solution to $xa + yb = c$, then the set of integer solutions to $xa + yb = c$ is exactly the set of $(x_0 - kb, y_0 + ka)$ for all integers k.

To see that no nonnegative combination of a, b makes $ab - a - b$, notice that $(b-1, -1)$ solves $xa + yb = ab - a - b$. So the set of all solutions is $(b - 1 - kb, -1 + ka)$. For solutions with $k \leq 0$, we have $y < 0$. For solutions with $k > 0$ we have $x < 0$. Therefore there are no nonnegative integer solutions. Ie, no nonnegative combination of a, b makes $ab - a - b$.

Now let N be any integer bigger than $ab - a - b$. Then the set

$$S = \{N + b, N + b - a, N + b - 2a, \ldots, N + b - (b - 1)a\}$$

is a set of b positive integers because

$$N + b - (b - 1)a > ab - a - b + b - (b - 1)a = 0.$$

Since $\gcd(a, b) = 1$, none of the integers in this set may be congruent mod b. (If the i-th and j-th terms are congruent mod b, then $b \mid (N + b - ia) - (N + b - ja)$ so $b \mid (j - i)a$ so $b \mid j - i$, which implies $i = j$). Therefore we get all the integers $\{0, \ldots, b - 1\}$ by reducing S mod b. In particular, there is an i-th term congruent to 0 mod b. This term is divisible by b, so there is some j such that $jb = N + b - ia$. Since $N + b - ia > 0$, we have $j > 0$ and in particular $(j - 1) \geq 0$. So $N = ia + (j - 1)b$ is a nonnegative linear combination of a and b that makes N.

(c) First, we claim that for $i = 1, \ldots, a - 1$, the smallest number congruent to ib modulo a is ib. Any smaller number is writable as $ax + by$ where $x, y \geq 0$ and $y < i$. This number is congruent to by modulo a, so if it were also congruent to bi, we would have $by \equiv bi \pmod{a}$. But a is coprime to b, so this implies $y \equiv i \pmod{a}$, a contradiction since $i < a$ implies i is the smallest positive number congruent to itself mod a.

Now, for any $i = 1, \ldots, a - 1$, write $ib = q_i a + r_i$ where $0 \leq r_i < a$ (this is the result of dividing ib by a and finding the quotient and remainder). Note that since by similar reasoning as above, $jb \equiv kb \pmod{a} \implies j = k$ when $0 \leq j, k < a$, so the r_i cycle through $1, \ldots, a - 1$ as i ranges from $1, \ldots, a - 1$.

The numbers congruent to r_i mod a that are not in $\langle a, b\rangle$ are $r_i, a + r_i, \ldots, (q_i - 1)a + r_i$, so there are precisely q_i such numbers. Hence, the genus of $\langle a, b\rangle$ is precisely $\sum_{i=1}^{a-1} q_i$ (we ignore the residue class of 0 modulo a since clearly all positive multiples of a are in $\langle a, b\rangle$). Now, use the fact that

$$\sum_{i=1}^{a-1} ib = \frac{ab(a-1)}{2} = a\sum_{i=1}^{a-1} q_i + \sum_{i=1}^{a-1} r_i = a\sum_{i=1}^{a-1} q_i + \frac{a(a-1)}{2},$$

which implies that

$$\sum_{i=1}^{a-1} q_i = \boxed{\frac{(a-1)(b-1)}{2}}.$$

5. (a) (i) $m(S) = 5$, $A(S) = \{0, 7, 11, 14, 18\}$. (ii) $m(S) = 3$, $A(S) = \{0, 7, 8\}$.

(b) Let's prove two statements: (1) every residue class mod m appears in $A(S)$ (and it appears as the smallest element of S in that residue class) and (2) no residue class mod m appears more than once in $A(S)$. It is obvious that the residue class 0 appears in $A(S)$, so we do not need to show that it appears in $A(S)$.

To see (1), let k be any nonzero residue class mod m, and let $x \in S$ such that $x \equiv k$ mod m. (Such an x exists because all numbers past some finite point are in S). By integer division, there is some q so that $0 < x - qm < m$. Since $x - qm$ is a positive integer smaller than the smallest positive integer in S, $x - qm \notin S$. The sequence $x, x - m, x - 2m, \ldots x - qm$ therefore starts with an element in S and ends up with an element not in S. So there is some point in the sequence where $x - im \in S$ and $x - (i + 1)m \notin S$. Then $x - im \in A(S)$ and therefore the residue class k mod m appears in $A(S)$. Furthermore, $x - im$ is the smallest element of S congruent to k because if $x - jm \in S$ for $j > i$ then $x - (i+1)m = (x - jm) + (j - i - 1)m \in S$, contradicting the fact that $x - (i+1)m \notin S$.

To see (2), assume for a contradiction that $x, y \in A(S)$ with $x \equiv y \mod m$ and $x < y$. Then $x \in S$. Since $x < y$, there is some integer $k \geq 0$ such that $y - m = x + km$. So $y - m \in S$ by additive closure. But $y - m \notin S$ because $y \in A(S)$. So we have a contradiction.

(c) It is sufficient to show that $A(S) \cup \{m\}$ generates S because removing 0 from a set of generators does not change what it generates.

Since $A(S) \cup \{m\} \subset S$, we have $\langle A(S) \cup \{m\} \rangle \subset S$.

To show the reverse inclusion, let $x \in S$. As in the previous proof, the sequence $x, x - m, x - 2m, \ldots$ eventually hits an element of $A(S)$. Thus x is an element of $A(S)$ plus some multiple of m. Ie, $x \in \langle A(S) \cup \{m\} \rangle$.

(d) We claim that the set

$$T = \bigcup_{a \in A(S)} T_a = \bigcup_{a \in A(S)} \{a - qm \mid q \geq 1,\ a - qm > 0\}$$

is exactly the set of positive integers not in S. Each element $a - qm$ is not in S because otherwise $a - m = (a - qm) + (q-1)m \in S$, contradicting the fact that $a - m \notin S$. Each positive integer x not in S is in T because eventually the sequence $x, x + m, x + 2m, \ldots$ hits some $a \in A(S)$.

Since no elements of $A(S)$ are congruent mod m, the sets T_a (for $a \in A(S)$) are disjoint. So we can count T by counting each of the sets T_a. The size of T_a is clearly equal to its corresponding Apéry coeffcient. Therefore

$$g(S) = \sum_{i=1}^{m-1} k_i.$$

(e) Obviously $\max_i((k_i - 1)m + i)$.

6. (a) Suppose $1 \leq i, j \leq m - 1$. Then $k_i m + i \in S$ and $k_j m + j \in S$ so $(k_i + k_j)m + (i + j) \in S$ by additive closure. Therefore the smallest element of S congruent to $i + j$ mod m is at most $(k_i + k_j)m + (i + j)$. If $i + j < m$, then the smallest element of S congruent to $i + j$ mod m is $k_{i+j}m + (i + j)$ so we get the inequality $k_{i+j} \leq k_i + k_j$. If $i + j > m$, then the smallest element of S congruent to $i + j$ mod m is $k_{i+j-m}m + (i + j - m)$ so we get the inequality $k_{i+j-m} - 1 \leq k_i + k_j$.

(b) Suppose $k_1, \ldots, k_{m-1}$ satisfy the inequalities given in part a. Let

$$A = \{0, k_1 m + 1, \ldots k_{m-1}m + m - 1\}.$$

Let $S = \langle A \cup \{m\} \rangle$. We claim that $A(S) = A$. By 5b, we can do this by showing that the smallest element congruent to i mod m is $k_i m + i$. So let $x \in S$ be the smallest element with $x \equiv i \mod m$. Then x is a positive linear combination of the generators in $A \cup \{m\}$. We can write the positive linear combination as follows:

$$x = (k_{j_1} m + j_1) + \ldots + (k_{j_n} m + j_n) + cm$$

for some sequence $1 \le j_1, \dots, j_n \le m-1$ (which might contain duplicate elements) and some positive integer c. If $c > 0$, then $x - m \in S$ is a smaller element with $x - m \equiv i \mod m$. So $c = 0$ and

$$x = (k_{j_1}m + j_1) + \ldots + (k_{j_n}m + j_n).$$

Reducing both sides mod m, we see that

$$i \equiv j_1 + \ldots + j_n \mod m.$$

Therefore $j_1 + \ldots + j_n = i + qm$ for some $q \ge 0$. Repeatedly applying the inequalities to $k_{j_1} + \ldots + k_{j_n}$, we get

$$k_{j_1} + \ldots + k_{j_n} + q \ge k_i.$$

Multiplying both sides by m and adding $j_1 + \ldots + j_n$ to both sides gives

$$k_{j_1}m + j_1 + \ldots + k_{j_n}m + j_n + qm \ge k_i m + j_1 + \ldots + j_n.$$

Move qm to the other side of the inequality and note that $j_1 + \ldots + j_n - qm = i$ to get

$$k_{j_1}m + j_1 + \ldots + k_{j_n}m + j_n \ge k_i m + i.$$

The left side of this inequality is simply x, so we have $x \ge k_i m + i$. Now x is the smallest element in S congruent to i, and $k_i m + i$ is an element in S congruent to i, so this forces $x = k_i m + i$ as desired.

(c) By problem 5 part e, the Frobenius number of S is $\max_i((k_i - 1)m + i)$; in order to have $(k_i - 1)m + i < 2m$ for all i, we must have all k_i equal to 1 or 2. Furthermore, since $1 + 1 \ge 2$ and $1 + 1 + 1 \ge 2$, any such choice of k_i automatically induces a valid Apéry set. By problem 5 part d, we have $g = \sum_{i=1}^{m-1} k_i$, so $g - (m - 1)$ of the k_is are 2s and the rest are 1s. Hence there are $\binom{m-1}{g-m+1}$ ways to choose those k_is to set to 2, and thus $\binom{m-1}{g-m+1}$ distinct such numerical semigroups.

(d) We prove by induction on g that we have $\sum_{m=1}^{g+1} \binom{m-1}{g-m+1} = F_{g+1}$, where the summand is understood to be 0 if $m - 1 < g - m + 1$, and $F_1 = F_2 = 1$. The base cases are easy to check. Suppose this is true for g and $g + 1$; then we have

$$\begin{aligned} F_{g+3} = F_{g+1} + F_{g+2} &= \sum_{m=1}^{g+1} \binom{m-1}{g-m+1} + \sum_{m=1}^{g+2} \binom{m-1}{g-m+2} \\ &= \sum_{m=1}^{g+1} \left(\binom{m-1}{g-m+1} + \binom{m-1}{g-m+2} \right) + \binom{g+1}{0} \\ &= \sum_{m=1}^{g+1} \binom{m}{g-m+2} + \binom{g+2}{0} = \sum_{m=1}^{g+2} \binom{m}{g-m+2} \\ &= \sum_{m=2}^{g+3} \binom{m-1}{g-m+3} = \sum_{m=1}^{g+3} \binom{m-1}{g-m+3} \end{aligned}$$

where we use Pascal's Identity to get from the second line to the third, and the fact that $\binom{1-1}{g-1+3} = 0$ (because $1 - 1 < g - 1 + 3$ for $g > -2$) for the last equality. So we are done.

7. We claim that $k_1, \ldots, k_{m-1}$ define an MED semigroup if and only if the constraints in problem 6, part a hold without equality: that is, for any $i, j \in \{1, \ldots, m-1\}$, we have $k_i + k_j > k_{i+j}$ if $i + j < m$, and $k_i + k_j + 1 > k_{i+j-m}$ if $i + j > m$.

   We first show that these conditions are necessary. Suppose one of the equalities from 6a holds. We write $B = \{A(S) - \{0\}\} \cup \{m\}$. We have two cases.

   Case 1: there exist $i, j \in \{1, \ldots, m-1\}$ with $i + j < m$ and $k_i + k_j = k_{i+j}$. Then $(k_i m + i) + (k_j m + j) = (k_i + k_j)m + i + j = k_{i+j}m + (i + j) \in B$, so $B \setminus \{k_{i+j}m + i + j\}$ has $m - 1$ elements and also generates S.

   Case 2: there exist $i, j \in \{1, \ldots, m-1\}$ with $i + j > m$ and $k_i + k_j + 1 = k_{i+j-m}$. Then $(k_i m + i) + (k_j m + j) = (k_i + k_j + 1)m + i + j - m = k_{i+j-m}m + i + j - m \in B$, so $B \setminus \{k_{i+j-m}m + i + j - m\}$ has $m - 1$ elements and also generates S.

   Next we show that these conditions are sufficient. Specifically, we claim that if these conditions are given, then for every $q \in \{1, \ldots, m-1\}$, an element of $\langle B \setminus \{k_q m + q\}\rangle$ that is congruent to $q \pmod m$ must be greater than $k_q m + q$, so all elements of B are needed to generate S.

   Take any $x \in \langle B \setminus \{k_q m + q\}\rangle$, and let $x = \sum_{i=1}^{n} a_i$ where $a_i \in B \setminus \{k_q m + q\}$ for all i. We induct on n. We have two base cases: $n = 1$ is obvious, and $n = 2$ is true by our given conditions: if $i + j \equiv q \pmod m$ for some $i, j \in \{1, \ldots, m-1\}$, then the inequalities tell us that $k_i m + i + k_j m + j > k_q m + q$.

   Now suppose for the sake of induction that our claim is true for n, and consider $\sum_{i=1}^{n+1} b_i \equiv q \pmod m$, $b_i \in B \setminus \{k_q m + q\}$. This can be written as $\sum_{i=1}^{n-1} b_i + b_n + b_{n+1}$. Let $b_n + b_{n+1} \equiv c \pmod m$ where $c \in \{1, \ldots, m-1\}$. Then, by the given conditions, either one of b_n, b_{n+1} is $k_c m + c$ or $b_n + b_{n+1} > k_c m + c$—but we have $b_n + b_{n+1} > k_c m + c$ in both cases. Hence $\sum_{i=1}^{n+1} b_i > \sum_{i=1}^{n-1} b_i + k_c m + c > k_q m + q$ by the inductive hypothesis. This completes the induction.

8. (a) They are, in the standard order,
   - $\langle 5, 6, 7, 8, 9\rangle, 4$
   - $\langle 4, 5, 7, 9\rangle, 5$
   - $\langle 4, 5, 7\rangle, 6$
   - $\langle 4, 5, 6\rangle, 7$
   - $\langle 3, 6, 7, 8\rangle, 5$
   - $\langle 3, 5\rangle, 7$
   - $\langle 2, 9\rangle, 7$

   (b) If you remove a generator from a numerical semigroup, then the result is still a numerical semigroup because no two elements in a numerical semigroup can sum to a generator (by our explicit algorithm for finding generators in problem 2). So all elements in the tree are valid numerical semigroups.

   Each numerical semigroup is its parent with one element removed, so the genus increases by exactly one at each level of the tree. So by induction, the level is equal to the genus.

   To see that we get every numerical semigroup, let S be any numerical semigroup. Let $a_1 < a_2 < \ldots < a_n$ be all the elements of $\mathbb{N}_0 - S$. I claim that the path starting at $\mathbb{N}_0$ and proceeding by removing each of the a_i's in order is a valid path through the tree. We can prove this by induction on the node in the path. The 0-th node $\mathbb{N}_0$ is at the root of the tree, which establishes the base case. For the inductive step, let $i \le n$ and assume that

$\mathbb{N}_0$, $\mathbb{N}_0 - \{a_1\}$, ..., $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ is a valid path through the tree. We need to show that one may move from $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ to $\mathbb{N}_0 - \{a_1, \ldots, a_i\}$ along the tree. Ie, we need to show that a_i is a generator of $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ that is greater than its Frobenius number. If a_i is not a generator, then some elements of $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ sum to it and therefore a_i cannot not be in S. Therefore a_i is a generator of $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$. a_i is obviously bigger than the Frobenius number because the Frobenius number is a_{i-1}. So we are done. We can reach all numerical semigroups through the tree.

The above path is the only path from the root to S because the Frobenius number constraint forces us to remove elements in increasing order. Therefore each numerical semigroup appears exactly once.

(c) $\langle 2, 2g+1 \rangle$ of genus g. To see this, we induct on g. The base case appears in the diagram. For the inductive step, suppose $\langle 2, 2g+1 \rangle$ of genus g is on the rightmost side of the tree. Its child is $\langle 2, 2g+1 \rangle - \{2g+1\}$ of genus $g+1$. It is easy to see that $\langle 2, 2g+1 \rangle - \{2g+1\} = \langle 2, 2(g+1)+1 \rangle$, completing the inductive step.

(d) $\langle g+1, \ldots, 2g+1 \rangle$ of genus g. To see this, we induct on g. The base case appears in the diagram. For the inductive step, suppose $\langle g+1, \ldots, 2g+1 \rangle$ of genus g is on the leftmost side of the tree. Its leftmost child is $\langle g+1, \ldots, 2g+1 \rangle - \{g+1\} = \{g+2, g+3, \ldots\}$. By applying the algorithm we described in the solution to 2, we get that this has generators $\{g+2, \ldots, 2(g+1)+1\}$, completing the inductive step.

9. Note: We use m below to denote $m(S) = m(S')$, which is valid because we are not in the leftmost branch of the semigroup tree. Therefore, the Frobenius number of S is greater than $m(S)$. Otherwise, S must be of the form $\langle g+1, \ldots, 2g+1 \rangle$ where g is the genus of S. We showed in 8d that such a semigroup must lie in the leftmost branch, and by 8b, this is the only place in which it occurs.

(a) Suppose S has Frobenius number F and the generator $F' > F$ is removed from S to give S'. Then our answers are: (i) all generators except F'; (ii) F'; (iii) $F'+m$ if $e(S) = e(S')$, otherwise none.

(b) Clearly all generators of S except F' are still generators of S', since if they could not be written in terms of other elements before, this will still not be possible when F' is removed. Now, note that a generator a of S' cannot be larger than $F'+m$, since otherwise we would have $a - m > F'$, hence $a - m \in S'$ and $a - m \neq 0$, hence a would be the sum of m and another nonzero element of S'. But if $a < F'+m$, and a is not the sum of two nonzero elements of S', note that a is also not the sum of F' and any nonzero element of S (since the smallest such element is m) and consequently a is also a generator of S. That is, all generators of S', except possibly $F'+m$, were also generators of S. Thus we have $e(S') = e(S)$ if $F'+m$ is a generator of S', and $e(S') = e(S) - 1$ otherwise.

10. (a) (i) 46. (ii) 12.

(b) The positive integers not in S, in terms of its Apéry coefficients, are just $\{1, m+1, \ldots, (k_1-1)m+1, 2, m+2, \ldots, (k_2-1)m+2, \ldots, m-1, m+m-1, \ldots, (k_{m-1}-1)m+m-1\}$. We have

$$\sum_{j=0}^{k_i-1} (jm+i) = m \cdot \sum_{j=0}^{k_i-1} j + ik_i = m \cdot \frac{k_i(k_i-1)}{2} + ik_i$$

so the weight is

$$\sum_{i=1}^{m-1}\left(m\cdot\frac{k_i(k_i-1)}{2}+ik_i\right).$$

(c) We extend the computations performed in Problem 4c. Recall that we wrote $ib = aq_i + r_i$ for $i = 1, \dots, a-1$ and $0 \le r_i < a$. r_i cycle through the numbers $1, \dots, a-1$ as i goes from 1 to $a-1$. Additionally, the numbers congruent to r_i mod a that are not in $\langle a, b\rangle$ are precisely the q_i numbers $r_i, a + r_i, \dots, (q_i - 1)a + r_i$.

From these preliminaries, we see that we wish to compute

$$\sum_{i=1}^{a-1}\sum_{j=0}^{q_i-1} aj + r_i = \sum_{i=1}^{a-1} q_i r_i + \frac{aq_i(q_i-1)}{2}.$$

Let $S = \sum_{i=1}^{a-1} \frac{1}{2}aq_i^2 + q_i r_i$, so that the desired sum is

$$S - \frac{a}{2}\sum_{i=1}^{a-1} q_i = S - \frac{a(a-1)(b-1)}{4}$$

by the computations in 4c.

Now, we write $(ib)^2 = (aq_i + r_i)^2 = a^2q_i^2 + 2aq_ir_i + r_i^2$ and sum over i to get

$$b^2\sum_{i=1}^{a-1} i^2 = \frac{b^2a(a-1)(2a-1)}{6} = 2aS + \sum_{i=1}^{a-1} r_i^2 = 2aS + \frac{a(a-1)(2a-1)}{6}.$$

This implies $S = \frac{(b^2-1)(a-1)(2a-1)}{12}$. Plugging back in, we see that the weight of $\langle a, b\rangle$ is

$$\begin{aligned}\frac{(b^2-1)(a-1)(2a-1)}{12} - \frac{a(a-1)(b-1)}{4} &= \frac{(a-1)(b-1)((b+1)(2a-1)-3a)}{12}\\ &= \boxed{\frac{(a-1)(b-1)(2ab-a-b-1)}{12}}.\end{aligned}$$

11. (a) Suppose $\mathbb{N}_0 \setminus S = \{1, 2, \dots, m-1, m+i_1, \dots, m+i_{g-m+1}\}$ with $i_a \in [1, m-1]$ for all a. We have

$$\begin{aligned} w &= 1 + 2 + \dots + m - 1 + (m + i_1) + \dots + (m + i_{g-m+1})\\ &\quad - (1 + 2 + \dots + m - 1 + m + \dots + g)\\ &= \sum_{a=1}^{g-m+1} (i_a - a + 1), \end{aligned}$$

which can be rearranged as

$$w - (g - m + 1) = \sum_{a=1}^{g-m+1} (i_a - a).$$

The $i_a - a$ are nonnegative because $i_1 \geq 1$ and $i_a > i_{a+1}$. They are non-decreasing since $i_{a+1} - (a+1) \geq i_a + 1 - (a+1) = i_a - a$. Finally, since $m - 1 \geq i_{g-m+1}$ and $i_{g-m+1} - (g - m + 1) \geq i_a - a$, we have $2m - 2 + g \geq i_a - a$. Thus each distinct choice of these i_a is associated with a unique partition of $w - (g - m + 1)$ into at most $g - m + 1$ parts, each of size at most $2m - 2 - g$. Furthermore, from any such partition $j_1 + \cdots + j_{g-m+1}$, where $0 \leq j_1 \leq \cdots \leq j_{g-m+1}$, it is easy to reconstruct S by setting $i_a = j_a + a$; the resulting i_a will be strictly increasing and bounded above by $m - 1$, as desired.

(b) Let N be the length of the hook associated with the top left square in the Ferrers-Young diagram of λ. Then the walk described in the hint has $N + 1$ total steps; denote the right steps by R and the up steps by U (so the first step is an R and the last step is a U), and number the steps from 0 to N. Note that each pair of steps $(i, i + j)$ where $i, j \geq 0$ and $0 \leq i, i + j \leq N$ such that step i is an R and step $i + j$ is a U corresponds uniquely to a hook of the diagram, with the length of the hook being j.

Suppose $a, b \in \mathbb{N}_0 \setminus H_\lambda$ but $a + b \in H_\lambda$, and choose i so that there is a hook of length $a + b$ beginning at step i and ending at step $i + a + b$ (that is, there is an R step at i and a U step at $i + a + b$). Since there is no hook of length a, step $i + a$ cannot be a U step (otherwise the pair $(i, i + a)$ would give a hook of length a); hence step $i + a$ is an R step. But then the pair $(i + a, i + a + b)$ starts with an R step and ends with a U step, hence gives a hook of length b, contradiction.

(c) Write $\mathbb{N}_0 \setminus S = \{n_1 < \cdots < n_g\}$, and consider the Ferrers-Young diagram whose walk along the bottom right edges (as described in the hint to part b) has $n_g + 1$ steps, with a U at steps $n_1, \ldots, n_g$ and an R everywhere else. By pairing the R at step 0 with the Us at steps $n_1, \ldots, n_g$, we can see that $n_1, \ldots, n_g \in H_\lambda$. We need to check that if $a \in S$, $a \notin H_\lambda$.

Suppose to the contrary that for some $a \in S$ there is a hook of length a; that is, there is i so that there is an R at step i and a U at step $i + a$. Then, by the construction of the walk, $i \in S$ and $i + a \in \mathbb{N}_0 \setminus S$. But we assumed that $a \in S$, so $i + a \in S$, a contradiction.