1. **Answer: 1500**

   Solution: There are $\binom{5}{3}$ possibilities for the range, so the answer is $10N$ where N is the number of surjective functions from $\{1,2,3,4,5\}$ to a given 3-element set. The total number of functions $\{1,2,...,5\} \to \{1,2,3\}$ is 3^5, from which we subtract $\binom{3}{2}$ (the number of 2-element subsets of $\{1,2,3\}$) times 2^5 (the number of functions mapping into that subset), but then (according to the Principle of Inclusion-Exclusion) we must add back $\binom{3}{1}$ (the number of functions mapping into a 1-element subset of $\{1,2,3\}$. Thus: $N = 3^5 - \binom{3}{2}(2^5) + \binom{3}{1}(1^5) = 150$. So $10N = 10(150) = \boxed{1500}$.

2. **Answer: $\mathbf{2, 6, 8, 10}$**

   Solution: We consider the value $\{k^n\} \pmod{12}$ for $k = 0, 1, \ldots, 11$ and $n = 1, 2, 3$.

$$n = 1 : \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$
$$n = 2 : \{0, 1, 4, 9, 4, 1, 0, 1, 4, 9, 4, 1\}$$
$$n = 3 : \{0, 1, 8, 3, 4, 5, 0, 7, 8, 9, 4, 11\}$$

   All values of k have period 1 or 2 (the rows for 2 and 8 continue $4, 8, 4, 8$, etc.) We can see that n^n cannot be congruent to $2, 6, 10$ when divided by 12 for $n > 1$, and hence cannot be congruent to $2, 6, 10$ at all. For n^n to be congruent to $8 \pmod{12}$, we would need either $n \equiv 2 \pmod{12}$ and $n \equiv 1 \pmod 2$ or $n \equiv 8 \pmod{12}$ and $n \equiv 1 \pmod 2$; this is impossible since a number which is 2 or $8 \pmod{12}$ must be even. All other remainders indeed occur; this can be checked by inspection, with the help of the Chinese Remainder Theorem. So our answer is $\boxed{2, 6, 8, 10}$.

3. **Answer: $\boldsymbol{\frac{9}{2}a_1a_2a_3}$**

   Solution: Let the x, y, and z intercepts of the plane be b_1, b_2, and b_3, respectively. The tetrahedron in question has volume $\frac{1}{6}b_1b_2b_3$. The equation of our plane must then be $\frac{x}{b_1} + \frac{y}{b_2} + \frac{z}{b_3} = 1$, since these three intercepts determine the plane. Therefore, we are minimizing $\frac{1}{6}b_1b_2b_3$ subject to the constraint $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} = 1$. By AM-GM, we get that

$$\frac{a_1a_2a_3}{b_1b_2b_3} \le \left(\frac{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3}}{3}\right)^3 = \frac{1}{27}.$$

   Therefore, $b_1b_2b_3 \ge 27a_1a_2a_3$ with equality if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{1}{3},$$

   which is a reasonable condition. Hence, the desired minimum volume is $\frac{27a_1a_2a_3}{6} = \boxed{\frac{9}{2}a_1a_2a_3}$.

4. **Answer: 811641**

   Solution: By taking tangents to both sides we have

$$\frac{1}{k} = \frac{1/x + 1/y}{1 - 1/xy} = \frac{x+y}{xy-1},$$

   so $xy = k(x+y) + 1$, $(x-k)(y-k) = k^2 + 1$. For $k = 2012$ the second largest factor of $k^2 + 1$ is $(k^2+1)/5 = 809629$ and thus second largest integer y is $k + 809629 = \boxed{811641}$.

5. **Answer: $\boldsymbol{\frac{9\sqrt{3}}{8}}$**

   Solution: By symmetry, $E[(B_iA_{i+1}B_{i+1})] = E[(B_jA_{j+1}B_{j+1})]$ for all integers i and j. Therefore, applying linearity of expectation, the expected area of $B_1B_2B_3B_4B_5B_6$ is equal to the area of $A_1A_2A_3A_4A_5A_6$ minus six times the expected area of $B_1A_2B_2$. Since the lengths of $\overline{B_1A_2}$ and $\overline{B_2A_2}$ are independent, this expectation is equal to $(\frac{1}{2}\sin 120°)E[B_1A_2]E[B_2A_2]$. It is easy to see that $E[B_1A_2] = E[B_2A_2] = 1/2$, so

$E[(A_1B_2A_2)] = \frac{\sqrt{3}}{16}$. The area of a unit regular hexagon is $6(\frac{\sqrt{3}}{4})$, so our answer is $6(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{16}) = \boxed{\frac{9\sqrt{3}}{8}}$. Alternatively, since the area of $B_1B_2B_3B_4B_5B_6$ is linear in location of B_i for each i, and B_i are all independent, we can argue that the average cases comes when B_i all all midpoints.

6. **Answer: 2**

Using the partial fraction
$$\frac{1}{m(n+m+1)} = \frac{1}{n+1}\left(\frac{1}{m} - \frac{1}{n+m+1}\right)$$
we can sum the series in m first, then it is
$$\begin{aligned}\sum_{n,m=1}^{\infty} \frac{1}{nm(n+m+1)} &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}(\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n+1}) \\ &= \sum_{k\le n+1} \frac{1}{kn(n+1)} = \sum_{k \le n+1} \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+1}\right).\end{aligned}$$

Now we sum this series in k first. For $k = 1$, n starts from 1, and for other values of k, n starts from $k-1$. Noting that each of these sums telescope,
$$\begin{aligned}&= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=k-1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = 1 + \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 + 1 = \boxed{2}.\end{aligned}$$

7. **Answer: 1161**

Solution: There are $3^4 = 81$ different cards. Any choice of two cards determines a unique line, since if we know a_i and b_i, c_i must equal a_i, b_i if $a_i = b_i$ and must equal neither if $a_i \neq b_i$. If we choose two distinct cards, we produces $\frac{1}{3}\binom{81}{2}$ distinct lines since each line can be generated by 3 pairs of cards. To this, we add 81 lines that are generated by choosing the same card three times (notice that if two cards in a line are the same the last card must also be the same). Therefore, our final answer is $\frac{1}{3}\binom{81}{2} + 81 = 27 \cdot 40 + 81 = \boxed{1161}$.

8. **Answer: $e^{e^{100}} - e$**

Solution: Let $x(t)$ and be the position of the ant. Since the length of the rubber band equals $100t + 100e$ and the rubber band stretches uniformly, any point on the rubber band will have velocity $\frac{x(t)}{100t+100e}100$. Thus, the total velocity of the bug equals $x'(t) = \frac{x(t)}{100t+100e}100 + \frac{1}{\ln t+e}$. Dividing by $t+e$ and rearranging yields $\frac{d}{dt}(\frac{x(t)}{t+e}) = \frac{1}{(t+e)\ln t+e}$. This integrates to $x(t) = (t+e)C_1 + (t+e)\ln(\ln(t+e))$, where $C_1 = 0$ from the initial conditions. The ant reaches the sadistic child when $x(t) = 100t + 100e$, at which point $\ln(\ln(t+e)) = 100$ or equivalently $t = \boxed{e^{e^{100}} - e}$.

9. **Answer: 1989**

Solution: For notational convenience, let ℓ be the line passing through the origin and $(1984, 2012)$.

First, note that a point P can only be nearby the give line segment if its x-coordinate is between 0 and 1984, inclusive. If the x-coordinate of P is negative, its distance to any point on the line segment is less than the distance between that point on the line segment and the point one unit to the right of P; if P has x-coordinate greater than 1984, take the point one unit to the left. This inequality holds due to the creation of a right or obtuse angle between P, the point next to P, and any point on the line segment (the edge case where the three points are collinear remains, but this is easily checked separately).

Now, fix a value $x \in \{0, 1, \ldots, 1984\}$, and let S be the set of lattice points with that x coordinate. Let Q be the point on the line segment with this x-coordinate. Note that for any $P \in S$, the distance between

P and the line segment is either the distance from P to ℓ or the distance from P to one of the endpoints of the line segment. In the latter case, since there always exists a cardinal direction to move closer to a given point, P is not nearby our segment. Now consider the former case. By similar triangles, the distance between P and ℓ is proportional to PQ, and so the only P which could possibly be nearby the segment are the P closest to Q. There are two such points if the y-coordinate of Q has fractional part $1/2$, and one such point otherwise.

Finally, we show that all such points are in fact nearby: this relies on the fact that the slope of ℓ is greater than 1. Consider a point P that is no further from ℓ than any other lattice point with the same x-coordinate. We already know that its distance to Q, the point on the line with same x-coordinate, is less than or equal to $1/2$. Now draw the horizontal line through P, intersecting ℓ at R. Since the slope of ℓ is greater than 1, we have $PR < PQ \leq 1/2$, and so P is, out of all lattice points with the same y coordinate, the closest one to ℓ. Hence, it is a nearby point.

Now, we just need to count the number of nearby points. There are 1985 different valid choices of x-coordinate, and we must double-count all the ones for which the point on ℓ with that x-coordinate has y-coordinate with real part $1/2$. Since ℓ is given by

$$y = \frac{2012}{1984}x = \frac{503}{496}x,$$

this condition holds when $x \equiv 248 \pmod{496}$, so there are 4 such values in the relevant interval. Hence, report $\boxed{1989}$.

10. **Answer:** $\mathbf{\frac{1}{31752}}$

Solution: 1 must be the first number in the permutation, which happens with probability $\frac{1}{14}$. The probability for 2 to come before 4, 5, 8, 9 10, and 11 is $\frac{1}{7}$. Similarly, 3 coming before 6, 7, 12, 13, and 14 happens with probability $\frac{1}{6}$. 4 before 8 and 9 with probability $\frac{1}{3}$. 5 before 10 and 11 with probability $\frac{1}{3}$. 6 before 12 and 13 with probability $\frac{1}{3}$. 7 before 14 with probability $\frac{1}{2}$. All these events are independent, so the answer is the product of the above probabilities, or $\boxed{\frac{1}{31752}}$.

11. **Answer:** $\mathbf{(27, +\infty)}$

Solution: We make the following modifications to the numerator: since $xyz = 1$, we may multiply x^{-3}, y^{-3}, z^{-3} by $x^3y^3z^3$, and also add $-1 + x^3y^3z^3$. The numerator then factors as $-1 + x^3 + y^3 + z^3 - x^3y^3 - x^3z^3 - y^3z^3 + x^3y^3z^3 = (x^3-1)(y^3-1)(z^3-1)$. Similarly, the denominator factors as $(x-1)(y-1)(z-1)$, so that the expression can be rewritten as $(x^2+x+1)(y^2+y+1)(z^2+z+1)$. Dividing this by xyz and writing $z = 1/(xy)$, we find that this is in fact $\left(1 + x + \frac{1}{x}\right)\left(1 + y + \frac{1}{y}\right)\left(1 + xy + \frac{1}{xy}\right)$. For any $a > 0$, we can expand $(a-1)^2 \geq 0$ to obtain $a + 1/a \geq 2$, so the product has value at least $(1+2)(1+2)(1+2) = 27$. Equality cannot occur: this would require $x = y = z = 1$, making the original denominator zero. Everything larger than 27 can occur, however: we simply consider the special case $y = x$, when the expression reduces to $\left(1 + x + \frac{1}{x}\right)\left(1 + x + \frac{1}{x}\right)\left(1 + x^2 + \frac{1}{x^2}\right)$, a continuous function of x which is unbounded as $x \to +\infty$. Thus we answer $\boxed{(27, +\infty)}$.

12. **Answer:** $\mathbf{12 + 6\sqrt{3}}$

Solution: We begin by computing the angles of the triangle; repeated application of the Law of Cosines gives us that the angles of the triangle are $15°$, $60°$, and $105°$, with greatest common divisor of $15°$. Given any triangle whose vertices coincide with the vertices of a regular n-gon, the smallest possible angle in the triangle is $\frac{\pi}{n}$. Thus a dodecagon with side length $\sqrt{2}$ is the smallest regular polygon that satisfies our current constraints. The area of a regular n-gon with side length s is $\frac{n}{4}\cot\left(\frac{\pi}{n}\right)s^2$. Using the sine and cosine half-angle formulas, we find that $\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}$, $\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$, and therefore $\cot\left(\frac{\pi}{12}\right) = 2 + \sqrt{3}$. Therefore, the area of our regular dodacegon is $3\cot\left(\frac{\pi}{12}\right)(\sqrt{2})^2 = \boxed{12 + 6\sqrt{3}}$.

13. **Answer: 20**

Solution: Denote $\gcd(x, y)$ by (x, y). Let $\mathcal{P}$ represent the property of n such that $n \mid a^2b+1 \Rightarrow n \mid a^2+b$ for all $a, b \in \mathbb{N}$. Let $\mathcal{Q}$ represent the property of n such that $(a, n) = 1 \Rightarrow n \mid a^4 - 1$ for all $a \in \mathbb{N}$. We shall prove that they are equivalent.

Proof that $\mathcal{P} \Rightarrow \mathcal{Q}$: Let a be a positive integer with $(a, n) = 1$ and hence $(a^2, n) = 1$. By Bzout's identity, we can find $b \in \mathbb{N}$ such that $n \mid a^2b + 1$. By $\mathcal{P}$, $n \mid a^2 + b$. Then $a^4 - 1 = a^2(a^2 + b) - (a^2b + 1)$, so $n \mid a^4 - 1$.

Proof that $\mathcal{Q} \Rightarrow \mathcal{P}$: Let a, b be positive integers with $n \mid a^2b + 1$. Clearly $(a, n) = 1$, so $n \mid a^4 - 1$. Using $a^2(a^2 + b) = (a^4 - 1) + (a^2b + 1)$ and the fact that a and n are relatively prime, $n \mid a^2 + b$.

Now we wish to find all n with property $\mathcal{Q}$. If a is odd, we have $a^4 - 1 = (a^2 - 1)(a^2 + 1)$, $a^2 \equiv 1 \pmod 8$, and $a^2 + 1$ is even, so $16 \mid a^4 - 1$. If $(a, 3) = 1$, we have $a^2 \equiv 1 \pmod 3$, so $3 \mid a^4 - 1$. If $(a, 5) = 1$, we have $5 \mid a^4 - 1$ by Fermat's Little Theorem. This argument shows that $n \mid 240$ is sufficient.

To show $n \mid 240$ is necessary, suppose n has property $\mathcal{Q}$, and let $n = 2^a \cdot k$, where k is odd. If $k > 1$, then $(k - 2, n) = 1$, so by $\mathcal{Q}$ we conclude that $n \mid (k - 2)^4 - 1$. Then $k \mid (k - 2)^4 - 1$, but $(k - 2)^4 \equiv (-2)^4 \equiv 16 \pmod k$, so $k \mid 15$. Now, since $(11, n) = 1$, $n \mid 11^4 - 1$, then $2^a \mid 11^4 - 1$, resulting in $a \le 4$. Thus $n \mid 240$ is also necessary.

The number of natural numbers n such that property $\mathcal{P}$ holds is simply the number of positive integer divisors of 240, which is $(4 + 1)(1 + 1)(1 + 1) = \boxed{20}$.

14. **Answer: $(-1, -1/2)$**

Solution 1: Take $x = 1/n$. Using the Taylor expansion $\log(1 - x) = -x - x^2/2 - x^3/3 - \cdots$,

$$F(x) = e(1 - x)^{1/x} = \exp\left(1 + \frac{\log(1 - x)}{x}\right) = \exp\left(-\frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} \cdots\right).$$

Since $F(0) = 1$, when $\alpha = -1$ the limit can be represented as derivative of F as:

$$\lim_{n\to\infty} \frac{e\left(1 - \frac{1}{n}\right)^n - 1}{1/n} = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = F'(0) = -\frac{1}{2}.$$

15. **Answer: $e^{-1+\sqrt{1+2\ln(2)^2}}$**

Solution: First, note that for fixed a, b, m, and n, $f(a, b, m, n) = f(a, b, m/2, n) = f(a, b, m, n/2)$ because we can go from an optimal sheet folding in one case to an optimal sheet folding in another case by either folding or unfolding the sheet in half, which scales the sheet's folded area by the same amount as the box's cross-sectional area. This implies that we can tile the region $m \in (0, a)$, $n \in (0, a)$ with an infinite number of rectangles for which the expectation of f is the same inside each rectangle. In particular, the set R of these rectangles is the set of all rectangles bounded by points of the form $(\frac{a}{2^x}, \frac{a}{2^y})$ and $(\frac{a}{2^{x+1}}, \frac{a}{2^{y+1}})$ in mn-space for nonnegative integers x and y. The expectation of f over the whole region is the same as the expectation of f inside any one of these rectangles. Let us choose to examine the rectangle where we have $m \in [a/2, a)$, $n \in [a/2, a)$. Clearly, it is optimal to fold the sheet exactly once in the dimension where the sheet has length a. For the other dimension, we may assume that $b \in [a, 2a)$ because if $b \ge 2a$, we are forced to fold it anyways until $b < 2a$. Now, we only have to fold in the b dimension once if $b/2 < \max(m, n)$, and otherwise we must fold twice. Therefore, the expected value of f is equal to

$$\frac{4}{a^2}\int_{a/2}^{a}\int_{a/2}^{a} \frac{(a/2)(b/2)(1/2)^{H(b/2-\max(m,n))}}{mn}\, dm\, dn,$$

where $H(x)$ is the Heaviside Step Function (defined as $H(x) = 1$ if $x \ge 0$ and 0 otherwise).

We can compute this integral by noticing that it is equal to

$$\frac{4}{a^2}\left(\int_{a/2}^{a}\int_{a/2}^{a} \frac{ab/4}{mn}\, dm\, dn - \int_{a/2}^{b/2}\int_{a/2}^{b/2} \frac{ab/8}{mn}\, dm\, dn\right).$$

Hence, this problem relies on evaluating

$$\int_c^d \int_c^d \frac{1}{xy}\,dx\,dy = \int_c^d \frac{\ln(d)-\ln(c)}{y}\,dy = (\ln(d)-\ln(c))^2 = \ln(d/c)^2.$$

Plugging in, we get that the original integral equals

$$\frac{4}{a^2}((ab/4)\ln(2)^2 - (ab/8)\ln(b/a)^2) = \frac{b}{2a}(2\ln(2)^2 - \ln(b/a)^2).$$

Let $k = b/a$, so that the expectation we are maximizing is $g(k) = \frac{k}{2}(2\ln(2)^2 - \ln(k)^2)$ over the domain $k \in [1, 2)$. The derivative of g is

$$g'(x) = \frac{1}{2}(2\ln(2)^2 - \ln(k)^2) - \ln k = -\frac{1}{2}(\ln(k)^2 + 2\ln(k) - 2\ln(2)^2),$$

which we set to zero, which yields

$$\ln(k) = \frac{-2 \pm \sqrt{4 + 8\ln(2)^2}}{2} = -1 \pm \sqrt{1 + 2\ln(2)^2}.$$

We now need to check two final things. First, we must see if either of these solutions to $g'(k) = 0$ are in the interval $(1, 2)$. Only the positive solution to the above quadratic could possibly result in k being in greater than 1 to begin with. The easiest way to see that this gives us a value in the interval $(1, 2)$ is by noticing that $g(1) = g(2) = \ln(2)^2$ (since the $k = 1$ and $k = 2$ cases both result in the folded sheet having the same area, namely $a/2 \times a/2$, for all choices of m and n), so we are guaranteed a point with zero derivative in the interval $(1, 2)$ by Rolle's Theorem.

Additionally, we must check that this is a local maximum and not a minimum. We claim here that $g(k)$ is concave down on the interval $(1, 2)$, so what we have found is a local maximum. First, $\ln(k)^2$ is concave up on $(1, 2)$ because

$$\frac{d^2}{dk^2}\ln(k)^2 = \frac{2 - 2\ln(k)}{k^2},$$

which is positive when $\ln(k) < 1 \iff k < e$. Hence, $2\ln(2)^2 - \ln(k)^2$ is concave down in the same interval. It is also clearly decreasing. Finally, we have for generic functions f and g that if $f(x) = xg(x)$, then $f'(x) = g(x) + xg'(x)$ and $f''(x) = 2g'(x) + xg''(x)$, so on an interval where g is decreasing, concave down, and x is positive, then f is guaranteed to also be concave down. This scenario holds for our function $g(k)$, so it is concave down.

Hence, report $\boxed{e^{-1+\sqrt{1+2\ln(2)^2}}}$.