1. **Answer:** $\frac{3\pi}{10}$

   Solution: Let O be the center of the circle. Note that CO bisects AB, so the areas of $\triangle ACO$ and $\triangle BCO$ are equal. Hence, the desired difference in segment areas is equal to the difference in the areas of the corresponding sectors. The sector corresponding to $\widehat{AC}$ has area $\frac{2\pi}{5}$, and the sector corresponding to $\widehat{BC}$ has area $\frac{\pi}{10}$, so the desired difference is $\boxed{\frac{3\pi}{10}}$.

2. **Answer: 252**

   Solution: We can use the standard method of setting up a two-variable system and solving for the height of the trapezoid. However, since one base is half the length of the other, we may take a shortcut. Extend AB and CD until they meet at E. Clearly, BC is a midline of triangle EAD, so we have $EA = 2BA = 26$ and $ED = 2CD = 28$. The area of EAD is therefore four times that of a standard 13-14-15 triangle, which we know is $\frac{1}{2} \times 14 \times 12 = 84$ (since the altitude to the side of length 14 splits the triangle into 9-12-15 and 5-12-13 right triangles). The area of the trapezoid is $\frac{3}{4}$ the area of EAD by similar triangles, and is therefore $3 \times 84 = \boxed{252}$.

   A similar solution notices that after dropping perpendiculars from B and C to AD, we are left with a rectangle and two triangles that sum to a 13-14-15 triangle.

3. **Answer:** $\sqrt{3}/2$

   Solution: Let x be 1/4 the area of ABC, and y the area of a 60 degree sector of O_a minus x. Note that

   $$S_a = S_b = S_c = 3x + y, \quad S = x + 3y,$$

   so $S_a + S_b + S_c - S = 8x = 2|\triangle ABC| = \boxed{\sqrt{3}/2}$.

4. **Answer:** $\frac{503}{6}$

   Solution: Let (XYZ) denote the area of triangle XYZ.

   After a bit of angle-chasing, we can use SAS congruence to prove that $\triangle DEF \cong \triangle BFE$, so $EB \cong DF$ and therefore $AE \cong FC$. If we draw altitudes from E and F onto CD and AB, respectively, we note that $2AE = 2FC = DF = BE$, so $AE = \frac{1}{3}AB$.

   Next, note that $\triangle AEX \sim \triangle CDX$, so $\frac{CX}{AX} = \frac{CD}{AE} = 3$. Also, $\triangle CFY \sim \triangle AEY$, so $\frac{CY}{AY} = \frac{CF}{AE} = 1$. Hence, $XY = \frac{1}{4}AC \implies (EXY) = \frac{1}{4}(EAC)$.

   Finally, $AE = \frac{1}{3}AB \implies (EAC) = \frac{1}{3}(BAC)$. Since BAC is half the rectangle and therefore has area 1006, we get $(EXY) = \frac{1006}{12} = \boxed{\frac{503}{6}}$.

5. **Answer:** $\sqrt{6}/6$

   Solution: It is obvious that the sphere must be tangent to each face, because if not, then it can be moved so that it is tangent to four faces; now the radius can be increased until the sphere is tangent to the other four. Additionally, it is clear that the center of the sphere should be in the center of the octahedron.

   Now notice that the sphere must be tangent to the octahedron at the centroid of each face. This can be seen by symmetry. It is clear that it should be tangent somewhere along the median from one vertex to the opposite side, and this is true for all three medians, which meet at the centroid.

   Now we can proceed in a few ways. One way is to isolate one half of the octahedron i.e. a square-based pyramid. Slice this pyramid in half perpendicular to the square base and parallel to one of the sides of the square base. This slice will go through the medians of two opposite triangular faces, in addition to the center of the sphere itself. Hence, we get an isosceles triangle ABC with base $BC = 1$ and legs of length $\sqrt{3}/2$. O, the center of the sphere, is the midpoint of BC. If the altitude from O to AB intersects AB at D, then we have

   $$OD \times AB = AO \times BO,$$

since both equal twice the area of AOB, and so $DO = \frac{AO \times BO}{AB} = \frac{1/\sqrt{2} \times 1/2}{\sqrt{3}/2} = \boxed{\sqrt{6}/6}$.

Alternatively, note that our octahedron can be obtained by reflecting the region $x + y + z \leq 1/\sqrt{2}$, $x, y, z \geq 0$ by xy, yz, zx plane. The inscribing sphere has its center at origin, so its radius is the distance from the origin to the plane $x + y + z = 1/\sqrt{2}$, which is $1/\sqrt{6}$.

6. **Answer: $2\pi/3 + \sqrt{3} - 1$**

Solution:

First, it is clear that all of face $ABCD$ can be painted black. This has area 1.

Now we look at the other two visible faces. By symmetry, we only need to consider one of these faces, say $BCGF$. Unfold $BCGF$ along BC so that it is coplanar with $ABCD$, forming a rectangle $AF'G'D$ with width 1 and height 2. Now, it is clear that the region that can be painted on $BCGF$ is precisely the part of $BCG'F'$ that is at most two units away from A. Let a circle centered at A with radius two intersect DG' at X. Since $AX = 2$, $AD = 1$, and $AD \perp XD$, we conclude that $\angle DAX = \frac{\pi}{3} \implies \angle F'AX = \frac{\pi}{6}$. Letting $(P_1P_2 \ldots P_n)$ denote the area of the n-gon with vertices $P_1, \ldots, P_n$, the desired area equals

$$\text{area of sector } F'AX + (AXD) - (ABCD) = \frac{2^2\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Putting all this together, we get our final answer to be

$$1 + 2(\pi/3 + \frac{\sqrt{3}}{2} - 1) = \boxed{\frac{2\pi}{3} + \sqrt{3} - 1}.$$

7. **Answer: $\frac{6\sqrt{11}}{11}$**

Solution: We claim that the answer is equal to the inradius in general. Let $T_a = AB_aC_a$, $T_b = A_bBC_b$, $T_c = A_cB_cC$ be the smaller triangles cut by the tangents drawn to O. Also let D, E, F be the points of tangency between O and BC, CA, AB respectively. By considering the fact that tangents to O should have same length we have $AB_a + B_aC_a + C_aA = AE + AF$. If we sum this over all vertices, then we can see that the sum of the perimeters of T_a, T_b, T_c is the same as the perimeter of A. Then, the Principle of Similarity gives $r_a + r_b + r_c = r$ where r is inradius of ABC. It can be calculated by Heron's formula as

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\sqrt{11 \cdot 6 \cdot 3 \cdot 2}}{11} = \boxed{\frac{6\sqrt{11}}{11}}.$$

Alternatively, let h_a denote the height of the altitude from A to BC, and let r be the inradius of ABC. Since $\triangle ABC \sim \triangle AB_aC_a$ and since the altitude from A to B_aC_a has length $h_a - 2r$, we get

$$\frac{r_a}{r} = \frac{h_a - 2r}{h_a}.$$

Noticing that

$$r = \frac{(ABC)}{\frac{1}{2}(a+b+c)} = \frac{ah_a}{a+b+c},$$

we get

$$r_a = r - 2r\frac{a}{a+b+c}.$$

Applying the same reasoning to r_b and r_c, we can compute

$$r_a + r_b + r_c = 3r - 2r = r.$$

8. **Answer:** $\frac{45\sqrt{14}}{56}$

   Solution: Let the point where all three circles intersect be denoted as X.

   First, note that AO_bXO_c, BO_cXO_a, and CO_aXO_b are all rhombi. This helps us easily prove that $AO_b \parallel BO_a$. Since these segments are also congruent, we get that ABO_aO_b is a parellogram, and hence $AB \cong O_aO_b$. Similarly, $BC \cong O_bO_c$ and $CA \cong O_cO_a$.

   Note that r is the circumradius of triangle $O_aO_bO_c$ at point X. The above congruences show that r is therefore the circumradius of triangle ABC, which is computed as

$$\frac{abc}{4(ABC)} = \frac{3 \times 5 \times 6}{4\sqrt{7 \times 4 \times 2 \times 1}} = \boxed{\frac{45\sqrt{14}}{56}}.$$

9. **Answer:** $\frac{39}{5}$

   Solution: Note that $\angle ADB$ and $\angle CBD$ are supplementary. Therefore, we can extend AD past D to a new point C' such that $\triangle DBC \cong \triangle BDC'$ (alternatively, consider flipping $\triangle DBC$ over the altitude to $\overline{BD}$). Since $\angle ABD \cong \angle AC'B$, we have $\triangle ABD \sim \triangle AC'B$, and so

$$\frac{AC'}{AB} = \frac{AB}{AD} \implies AC' = \frac{AB^2}{AD} = \frac{64}{5}.$$

   Since $AC' = AD + DC'$, we get $DC' = BC = \frac{64}{5} - 5 = \boxed{\frac{39}{5}}$.

10. **Answer:** $4\sqrt{3}$

   Solution: We have $\angle ABC = \angle DAB = 120°$ and $\angle BCD = \angle CDA = 60°$ at the base, and the three "side" faces – ADE, BCF, and CDG – are all equilateral triangles. If those faces are folded down to the glass plate along AD, BC, and CD respectively, they will form (along with $ABCD$) a large equilateral triangle of side length 3. Let E_0, F_0, and G_0 be the vertices of this equilateral triangle corresponding to E, F and G, respectively.

   Observe that M, the midpoint of CD, is the centroid of $E_0F_0G_0$. As side ADE is folded along AD, which is perpendicular to E_0M, the projection E_1 of E (directly downwards onto the glass plate) still lies on E_0M. This also holds for the projections of F and G, so projections E_1, F_1, and G_1 of E, F, and G lie on E_0M, F_0M and G_0M respectively.

   Since $EFCD$ is a rectangle, E_1F_1CD is as well. Thus E_1D is perpendicular to E_0A. Furthermore, since E_0E_1 is perpendicular to AD, we can conclude that E_1 should be the center of triangle ADE_0. Symmetry gives $AE = DE = E_0E$, so AE_0DE should be a regular tetrahedron. A similar argument applies to BF_0CF.

   The next step is to figure out the location of G. As $EG = \sqrt{3}$, $ED = 1$, and $DG = 2$, it follows that $\angle DEG$ is right. Similarly $\angle CFG$ is also right, so plane EFG should be perpendicular to plane $EFCD$.

   Now we cut the whole object along the perpendicular bisector plane of AB and consider its cross-section along the plane. It will cut AB and EF along their midpoints N and P respectively. As $ABMP$ forms a regular tetrahedron of side length 1 and N is midpoint of AB, we have $NM = NP = \sqrt{3}/2$. Also $MG = \sqrt{3}$ and $\angle MPG$ is right. Let Q be the midpoint of MG; since right triangles are inscribed in semicircles, it follows that $PQ = MQ = \sqrt{3}/2$. Thus, NPM and QMP are congruent and NP and MG are parallel. From $MG = MG_0 = \sqrt{3}$ and $NP = NM = \sqrt{3}/2$, this gives similarity between NMP and MG_0G, and $GG_0 = 2PM = 2$. Therefore $DCGG_0$ also forms a regular tetrahedron.

   Since AE_0DE , BF_0CF, and CG_0DG are all regular tetrahedrons, we have three lines E_0E, F_0F, and G_0G meeting at a point X where $E_0F_0G_0X$ forms a regular tetrahedron of side length 3. Thus we finally demystified our object completely: it was obtained by cutting the regular tetrahedron $E_0F_0G_0X$ along planes EFG, ADE, BCF, CDG. Moreover we find that X is actually our point source, as it is also directly above M - both the midpoint of CD and the center of $E_0F_0G_0$ - and its height is $\sqrt{6}$, the same as that of point source. So the projection of the object to the glass plate will be exactly $E_0F_0G_0$, an equilateral triangle of side length 3. Hence the projection down to the wood plate will give an equilateral triangle of side length 4, and our answer is its area, $\boxed{4\sqrt{3}}$.