1. **Answer: 10**

   Solution: We know that this expression has to be a concave-up parabola (i.e. a parabola that faces upwards), and there is symmetry across the line $x = 3$. Hence, we conclude that the vertex of the parabola occurs at $x = 3$. Plugging in, we get $4 + 1 + 0 + 1 + 4 = \boxed{10}$.

2. **Answer: 22**

   Solution: Let a and b be the number of problems Alice and Bob solve, respectively. Then $ab = 3(a + b)$. Adding 9 to both sides and rearranging, $(a - 3)(b - 3) = 9$. The possible solutions are $(a, b) = (0, 0)$, $(4, 12)$, $(6, 6)$, and $(12, 4)$ which sum to $\boxed{22}$.

3. **Answer: $\frac{91}{136}$**

   Solution: We note

$$\prod_{n=2}^{k} \frac{n^3 - 1}{n^3 + 1} = \prod_{n=2}^{k} \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \left(\prod_{n=2}^{k} \frac{n-1}{n+1}\right)\left(\prod_{n=2}^{k} \frac{n^2+n+1}{n^2-n+1}\right)$$

   Each product telescopes, yielding $\frac{1\cdot 2}{k\cdot(k+1)} \cdot \frac{k^2+k+1}{3}$. Evaluating at $k = 16$ yields $\boxed{\frac{91}{136}}$.

4. **Answer: 2047**

   Solution: The possible values of b are precisely the powers of two not exceeding 2012 (including $2^0 = 1$). The following proof uses the fact that the zeroes of sine and cosine are precisely numbers of the form $t\pi$ and $(t + 1/2)\pi$, respectively, for t an integer.

   Suppose b is not a power of 2. Then it can be written as $2^m(1 + 2k)$ for $m \geq 0$, $k > 0$. Since $2^m < b$, by assumption f must have a root at 2^m. But then f must have a root at b, too:
   • If $\sin(2^m a) = 0$, then $2^m a = t\pi$ for some integer t, so $\sin(ba) = \sin((1 + 2k)2^m a) = \sin((1 + 2k)t\pi) = 0$ and b is a root of f.
   • If $\cos(2^m a) = 0$, then $2^m a = (t + 1/2)\pi$ for some integer t so

$$\cos(ba) = \cos((1 + 2k)2^m a) = \cos((1 + 2k)(t + 1/2)\pi) = \cos((t + k + 2kt + 1/2)\pi) = 0$$

   and thus b is a root of f.
   This is a contradiction, so b can only be a power of 2.

   For each b of the form 2^m, we can construct an f that works by using cosine terms to cover integers preceding b and sine terms thereafter:

$$f(x) = \left(\prod_{i=1}^{m} \cos(\pi x/2^i)\right)\left(\prod_{j=b+1}^{2012} \sin(\pi x/j)\right)$$

   has a root at every positive integer at most 2012 except b.

   Hence, our final answer is $1 + 2 + 4 + ... + 1024 = 2048 - 1 = \boxed{2047}$.

5. **Answer: −125**

   Solution: If the three roots of f are r_1, r_2, r_3, we have $f(x) = x^3 - (r_1+r_2+r_3)x^2 + (r_1r_2+r_1r_3+r_2r_3)x - r_1r_2r_3$, so $f(-1) = -1 - (r_1 + r_2 + r_3) - (r_1r_2 + r_1r_3 + r_2r_3) - r_1r_2r_3$. Since $r_1r_2r_3 = 64$, the arithmetic mean-geometric mean inequality reveals that $r_1 + r_2 + r_3 \geq 3(r_1r_2r_3)^{1/3} = 12$ and $r_1r_2 + r_1r_3 + r_2r_3 \geq 3(r_1r_2r_3)^{2/3} = 48$. It follows that $f(-1)$ is at most $-1 - 12 - 48 - 64 = \boxed{-125}$. We have equality when all roots are equal, i.e. $f(x) = (x - 4)^3$.

6. **Answer: −24**

   Solution: Consider the polynomial $Q(x) = P(x) - 3$. Q has roots at $x = 2$ and $x = 3$. Moreover, since these roots are maxima, they both have multiplicity 2. Hence, Q is of the form $a(x - 2)^2(x - 3)^2$, and so $P(x) = a(x-2)^2(x-3)^2+3$. $P(1) = 0 \implies a = -\frac{3}{4}$, allowing us to compute $P(5) = -\frac{3}{4}(9)(4)+3 = \boxed{-24}$.

7. **Answer: −14**

Solution: By Viéta's Formulas, we have that $f(0) = -a_1b_1c_1$ and $g(0) = -a_2b_2c_2$. Additionally, $(a - \frac{1}{b})(b - \frac{1}{c})(c - \frac{1}{a}) = -3$ and $(a - \frac{1}{b}) + (b - \frac{1}{c}) + (c - \frac{1}{a}) = 5$. Expanding the first expression yields $-3 = abc - \frac{1}{abc} - ((a+b+c) - (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})) = abc - \frac{1}{abc} - 5$. This is equivalent to $(abc)^2 - 2(abc) - 1 = 0$, so $abc = 1 \pm \sqrt{2}$. It follows that $f(0)^3 + g(0)^3 = -(1+\sqrt{2})^3 - (1-\sqrt{2})^3 = \boxed{-14}$.

8. **Answer: (1,1,1) (1,3,3) (3,1,3) (3,3,1)**

Solution: Rearranging the given equality yields $xyz - 2(xy + xz + yz) + 4(x + y + z) - 8 = -1$. But the left side factors as $(x-2)(y-2)(z-2)$. Since all quantities involved are integral, we must have each factor equal to ± 1. It is easy to verify that the only possibilities for (x, y, z) are those listed.

9. **Answer: $\sqrt{5}$**

Solution: We shall use the formula $S_{\triangle P_1OP_2} = \frac{1}{2}|z_1||z_2|\sin\theta$, where θ is the angle between z_1 and z_2. Solving for z_2 using the quadratic equation, we obtain $z_2 = \frac{1 \pm \sqrt{5}i}{2} z_1$. From this relationship we see that $\tan\theta = \sqrt{5}$, so $\sin\theta = \frac{\sqrt{5}}{\sqrt{6}}$; also, $|z_2| = \frac{\sqrt{6}}{2}|z_1|$. Thus

$$S_{\triangle P_1OP_2} = \frac{1}{2}|z_1|\left(\frac{\sqrt{6}}{2}|z_1|\right)\left(\frac{\sqrt{5}}{\sqrt{6}}\right) = \frac{\sqrt{5}}{4}|z_1|^2.$$

Now, we note that $|z_1|$ must be 2. There are many ways to see this. Geometrically, $z_1 - 2$ and $z_1 + 2$ form a right triangle with the origin because they are $\frac{\pi}{2}$ apart. z_1 is then the median from the origin to the hypotenuse, so its magnitude is equal to half the length of the hypotenuse, which is $(z_1+2) - (z_1-2) = 4$. Or we can set $\frac{z_1-2}{z_1+2} = bi$. Solving, we obtain $z_1 = \frac{-2(b-i)}{b+i}$ which satisfies $|z_1| = 2$.

Therefore, $S_{\triangle P_1OP_2} = \frac{\sqrt{5}}{4} \cdot 2^2 = \boxed{\sqrt{5}}$.

10. **Answer: $\mathbf{\frac{2013}{4025}}$**

Solution: To simplify notation, define $Y_n = \lceil \log_{2n} X_n \rceil$.

We begin by computing the probability that Y_n is odd. $Y_n = -1$ if $-2 < \log_{2n} X_n \le -1$, or $\frac{1}{(2n)^2} < X_n \le \frac{1}{2n}$. Similarly, $Y_n = -3$ if $\frac{1}{(2n)^4} < X_n \le \frac{1}{(2n)^3}$, and so on. Adding up the lengths of these intervals, we see that the probability that Y_n is odd is

$$\sum_{k=1}^{\infty} \frac{1}{(2n)^{2k-1}} - \frac{1}{(2n)^{2k}} = \frac{\frac{1}{2n}(1 - \frac{1}{2n})}{1 - \frac{1}{(2n)^2}} = \frac{\frac{1}{2n}}{(1 + \frac{1}{2n})} = \frac{1}{2n+1}.$$

Armed with this fact, we are now ready to solve the problem. One way to continue would be to note that the probability that Y_1 is even is $2/3$, the probability that $Y_1 + Y_2$ is even is $3/5$, the probability that $Y_1 + Y_2 + Y_3$ is even is $4/7$ and to show by induction that the probability that $Y_1 + \cdots Y_n$ is even is $\frac{n+1}{2n+1}$. Below, we present an alternate approach.

Note that $Y_1 + Y_2 + \cdots + Y_{2012}$ is even if and only if $(-1)^{Y_1+Y_2+\cdots+Y_{2012}} = 1$. Rewrite $(-1)^{Y_1+Y_2+\cdots+Y_{2012}}$ as $(-1)^{Y_1}(-1)^{Y_2}\cdots(-1)^{Y_{2012}}$, and note that because the Y_n are independent,

$$E\left[(-1)^{Y_1}(-1)^{Y_2}\cdots(-1)^{Y_{2012}}\right] = E[(-1)^{Y_1}]E[(-1)^{Y_2}]\cdots E[(-1)^{Y_{2012}}], \tag{1}$$

where E denotes the expected value of the quantity. But $E[Y_n] = (+1)\cdot P(Y_n \text{ is even}) + (-1)\cdot P(Y_n \text{ is odd})$. We computed earlier that the probability that Y_n is odd is $\frac{1}{2n+1}$, so $E[Y_n] = \frac{2n-1}{2n+1}$ and product in (1) is $\frac{1}{3} \cdot \frac{3}{5} \cdots \frac{4023}{4025}$, which telescopes to yield $\frac{1}{4025}$. Let p be the probability that $Y_1 + Y_2 + \cdots Y_{2012}$ is even. We just found that $(+1)(p) + (-1)(1-p) = \frac{1}{4025}$, which we can solve to obtain $p = \boxed{\frac{2013}{4025}}$.