1. **Answer:** $\mathbf{2+2\sqrt{2}}$

   The path made generates a regular octagon with side length 1, since the exterior angle of the octagon is 45 deg. Notice that by inscribing the octagon in a square of side length $1+\sqrt{2}$, we can easily calculate that the octagon has area $(1+\sqrt{2})^2 - 4\left(\frac{1}{4}\right) = 2+2\sqrt{2}$.

2. **Answer:** $\mathbf{\frac{33\sqrt{3}}{28}}$

   Looking at cyclic quadrilaterals $ABCD$ and $ACDF$ tells us that $m\angle ACD = m\angle ADC$, so $\triangle ACD$ is equilateral and $m\angle DEA = 120°$. Now, if we let $m\angle EAD = \theta$, we see that $m\angle CAB = 60° - \theta \implies m\angle ACB = \theta \implies \triangle AED \cong \triangle CBA$. Now all we have to do is calculate side lengths. After creating some $30° - 60° - 90°$ triangles, it becomes evident that $AC = \sqrt{3}$. Now let $AB = x$, so $BC = 2x$. By applying the Law of Cosines to triangle ABC, we find that $x^2 = \frac{3}{7}$. Hence, the desired area $(ABCDE) = (ACD) + 2(ABC) = \frac{(\sqrt{3})^2\sqrt{3}}{4} + 2\cdot\frac{1}{2}(x)(2x)(\sin 120°) = \frac{33\sqrt{3}}{28}$.

3. **Answer: 15**

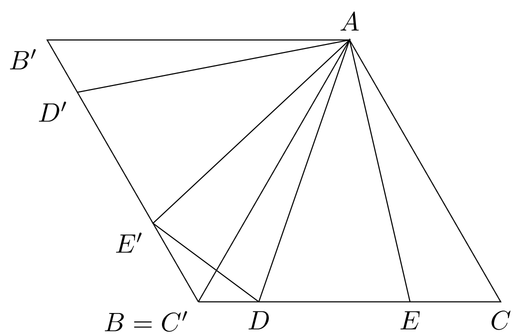
   After some angle chasing, we find that $m\angle DBF = m\angle DFB = 75°$, which implies that $DF = DB$. Hence the desired perimeter is equal to $AF - BF + AE + FE = 20 - BF + FE$.

   By the law of sines, $\frac{FE}{\sin 30°} = \frac{10}{\sin 75°} \implies FE = \frac{5}{\frac{\sqrt{6}+\sqrt{2}}{4}} = 5\sqrt{6} - 5\sqrt{2}$.

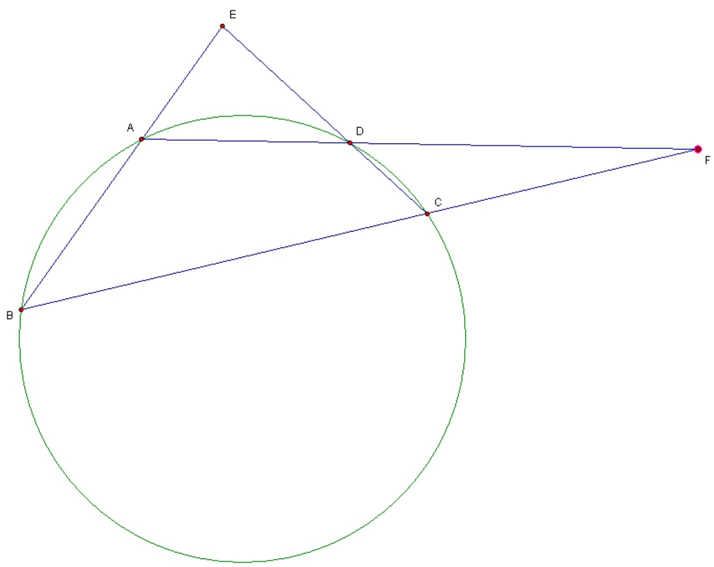
   Now, to find $\overline{BF}$, draw the altitude from O to $\overline{AB}$ intersecting $\overline{AB}$ at P. This forms a $30° - 60° - 90°$ triangle, so we can see that $AP = 5\sqrt{3}/2 = \frac{10-BF}{2} \implies BF = 10 - 5\sqrt{3}$. Hence, the desired perimeter is $20 + (5\sqrt{6} - 5\sqrt{2}) - (10 - 5\sqrt{3}) = 10 - 5\sqrt{2} + 5\sqrt{3} + 5\sqrt{6}$, so the answer is $10 - 5 + 5 + 5 = 15$.

4. **Answer:** $\mathbf{5+\sqrt{19}}$

   Rotate the figure around A by $60°$ so that C coincides B. Let B', C', D', E' be the points corresponding to B, C, D, E in the rotated figure. Since $\angle E'AD = \angle E'AC' + \angle C'AD = \angle EAC + \angle BAD = 30° = \angle EAD$, $E'A = EA$ and $DA = D'A$, one has $E'D = ED$. So $BC = BD + DE + EC$ can be found if we know $E'D$. But $E'D = \sqrt{E'B^2 + BD^2 - 2\cdot E'B\cdot BD\cdot\cos 120°} = \sqrt{19}$, so $BC = 2 + \sqrt{19} + 3 = 5 + \sqrt{19}$.

5. **Answer:** $\mathbf{10\sqrt{2}}$



We have $\triangle ADE \sim \triangle CBE$, and their length ratio is $AD : CB = 1 : 2$. Let $AE = p$ and $DE = q$. Then we have $AB = BE - AE = 2DE - AE = 2q - p$ and $CD = 2p - q$. Solving for p and q, we have $p = 4$ and $q = 5$. Similarly we have $FC = 8$ and $FD = 10$. Let $\angle B = \theta$. Then $\angle FDE = \pi - \theta$. Apply the Law of Cosines to $\triangle EBF$ to get

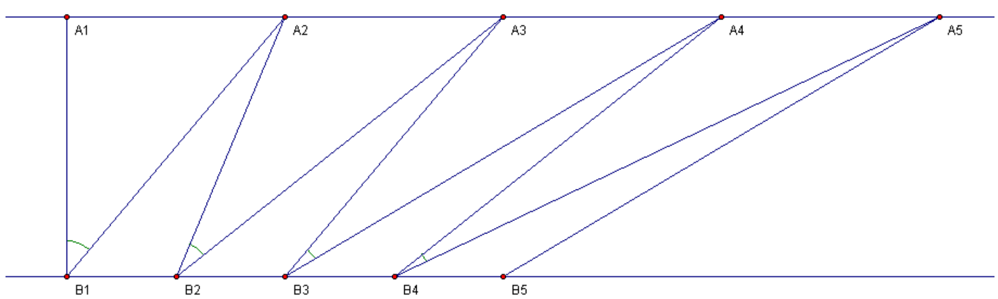
$$EF^2 = BE^2 + BF^2 - 2BE \cdot BF \cdot \cos\theta = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos\theta = 500 - 400\cos\theta$$

and to $\triangle EDF$ to get

$$EF^2 = DE^2 + DF^2 + 2 \cdot DE \cdot DF \cos\theta = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos\theta = 125 + 100\cos\theta.$$

Solving for EF^2, we get $EF^2 = 200$.

6. **Answer: $\boldsymbol{\pi - \tan^{-1}(\frac{1}{d})}$ (or $\boldsymbol{\pi/2 + \tan^{-1} d}$) or other equivalent form**



Construct points $C_1, C_2, C_3, \cdots$ on l_1 progressing in the same direction as the A_i such that $C_1 = A_1$ and $C_nC_{n+1} = 1$. Thus we have $C_1 = A_1$, $C_3 = A_2$, $C_5 = A_3$, etc., with $C_{2n-1} = A_n$ in general. We can write $\angle A_iB_iA_{i+1} = \angle C_{2i-1}B_iC_{2i+1} = \angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1}$. Observe that $\triangle C_iB_iC_k$ (for any k) is a right triangle with legs of length d and $k - i$, and $\angle C_iB_iC_k = \tan^{-1}\frac{k-i}{d}$. So $\angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1} = \tan^{-1}\frac{i+1}{d} - \tan^{-1}\frac{i-1}{d}$. The whole sum is therefore

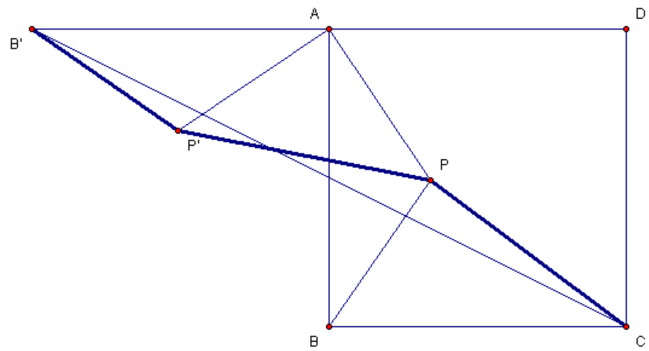
$$\sum_{i=1}^{\infty}\left(\tan^{-1}\frac{i+1}{d} - \tan^{-1}\frac{i-1}{d}\right)$$

which has nth partial sum

$$\tan^{-1}\frac{n+1}{d} + \tan^{-1}\frac{n}{d} - \tan^{-1}\frac{1}{d}$$

so it converges to $\pi - \tan^{-1}\frac{1}{d}$.

7. **Answer:** $\boldsymbol{\sqrt{5}}$



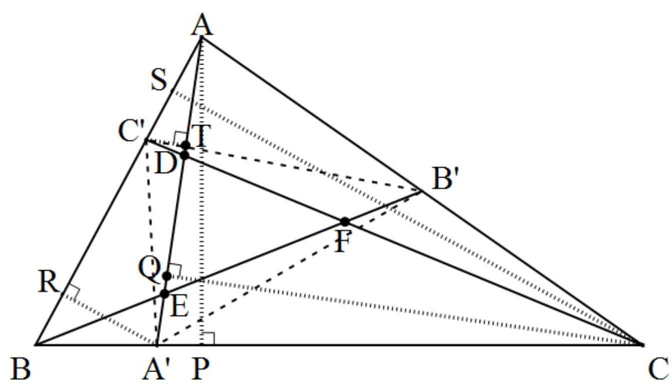
Rotate triangle APB around A by 90 degrees as in the given figure. Let P' and B' be the rotated images of P and B respectively. Then we have $B'P' = BP$, $P'P = \sqrt{2}AP$ so

$$\sqrt{2}AP + BP + CP = CP = PP' + P'B' \leq CB' = \sqrt{5}.$$

8. **Answer:** $\boldsymbol{\frac{\pi}{6}}$

Consider the cube to be of side length 2 and divide the answer by 4 later. Set the coordinates of the vertices of the cube to be $(\pm1, \pm1, \pm1)$. Then the plane going through an equilateral triangle can be described by the equation $x+y+z = 1$. The distance to the plane from the origin is $\frac{1}{\sqrt{3}}$, as $(1/3, 1/3, 1/3)$ is the foot of the perpendicular from $(0, 0, 0)$. Thus the radius of the circle is $\sqrt{1 - (\frac{1}{\sqrt{3}})^2} = \sqrt{\frac{2}{3}}$, so the area is $\frac{2}{3}\pi$. In the case of the unit cube we should divide this by 4 to get the answer $\frac{\pi}{6}$.

9. **Answer:** $\boldsymbol{\frac{49}{390}}$

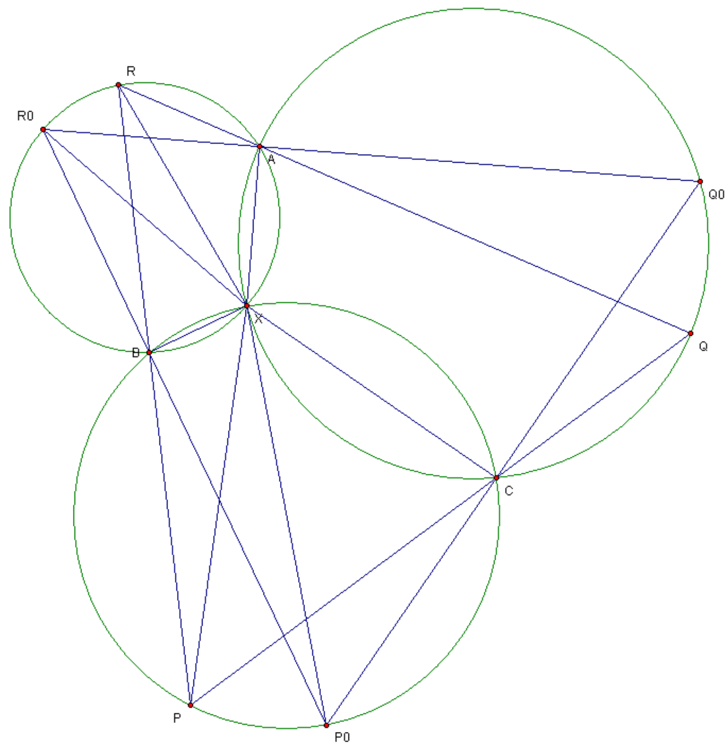


First we shall find $\frac{area(\triangle ADC)}{area(\triangle ABC)}$: Since $\triangle A'RB \sim CC'B$ and $A'B = \frac{1}{5}BC$, it follows that $A'R = \frac{1}{5}CS$. Then $area(\triangle AA'C') = \frac{1}{2}A'R \times AC' = \frac{1}{2}(\frac{1}{5}CS)(\frac{1}{3}AB) = \frac{1}{15}(\frac{1}{2}CS \times AB) = \frac{1}{15}area(\triangle ABC)$. Similarly, $area(\triangle AA'C) = \frac{1}{2}AP \times A'C = \frac{1}{2}AP(\frac{4}{5}BC) = \frac{4}{5}area(\triangle ABC)$. So $\frac{area(\triangle AA'C')}{area(\triangle AA'C)} = \frac{1}{12}$. Since $\triangle AA'C'$ and $\triangle AA'C$ share the same base, $\frac{C'T}{QC} = \frac{1}{12}$. Since $\triangle C'TD \sim \triangle CQD$, $\frac{C'D}{CD} = \frac{1}{12}$. Using similar arguments, since $AC' = \frac{1}{3}AB$, $area(\triangle AC'C) = \frac{1}{3}area(\triangle ABC)$. Since $CD = \frac{12}{13}C'C$, $area(\triangle ADC) = \frac{12}{13}area(\triangle AC'C) = \frac{12}{13} \times \frac{1}{3}area(\triangle ABC)$. Using the same technique, we can find $\frac{area(\triangle BFC)}{area(\triangle ABC)}$ and $\frac{area(\triangle AEB)}{area(\triangle ABC)}$. We will just briefly outline the remaining process: $\frac{area(\triangle CC'B')}{area(\triangle CC'B)} = \frac{\frac{1}{2}\times\frac{1}{3}}{\frac{2}{3}} = \frac{1}{4}$. So $\frac{B'F}{BF} = \frac{1}{4}$. Then $area(\triangle BB'C) = \frac{1}{2}area(\triangle ABC)$, so $area(\triangle BFC) = \frac{4}{5}area(\triangle BB'C) = \frac{2}{5}area(\triangle ABC)$.

Likewise, $\frac{area(\triangle BB'A')}{area(\triangle BB'A)} = \frac{\frac{1}{5}\times\frac{1}{2}}{\frac{1}{2}} = \frac{1}{5}$. So $\frac{A'E}{AE} = \frac{1}{5}$. Then $area(\triangle AA'B) = \frac{1}{5}area(\triangle ABC)$, so $area(\triangle AEB) = \frac{5}{6}area(\triangle AA'B) = \frac{1}{6}area(\triangle ABC)$.

Then $\frac{area(\triangle DEF)}{area(\triangle ABC)} = 1 - \frac{area(\triangle ADC)}{area(\triangle ABC)} - \frac{area(\triangle BFC)}{area(\triangle ABC)} - \frac{area(\triangle AEB)}{area(\triangle ABC)} = 1 - \frac{12}{39} - \frac{2}{5} - \frac{1}{6} = \frac{49}{390}$.

10. **Answer: $\mathbf{2\sqrt{43}}$**



We claim that in general, the answer is $\sqrt{\frac{2}{3}(a^2+b^2+c^2+4\sqrt{3}S)}$, where S is the area of ABC.

Suppose that PQR is an equilateral triangle satisfying the conditions. Then $\angle BPC = \angle CQA = \angle ARB = 60^\circ$. The locus of points satisfying $\angle BXC = 60^\circ$ is part of a circle O_a. Draw O_b and O_c similarly. These three circles meet at a single point X inside the triangle, which is the unique point satisfying $\angle BXC = \angle CXA = \angle AXB = 120^\circ$. Then the choice of P on O_a determines Q and R: those two points should also be on O_b and O_c respectively, and line segments PCQ and PBR should form sides of the triangle. Now one should find the maximum of PQ under these conditions. Note that $\angle BPX$ and $\angle BRX$ do not depend on the choice of P, so triangle PXR has the same shape regardless of our choice. In particular, the ratio of PX to PR is constant, so PR is maximized when PX is the diameter of O_a. This requires PQ, QR, RP to be perpendicular to XC, XA, XB respectively.

From this point there may be several ways to calculate the answer. One way is to observe that $PQ = \frac{2}{\sqrt{3}}(AX+BX+CX)$ by considering $(PQR) = (PXQ) + (QXR) + (RXP)$. $AX+BX+CX$ can be computed by the usual rotation trick for the Fermat point: rotate $\triangle BXA$ 60° around B to $\triangle BX'A'$. Observe that $\triangle BXX'$ is equilateral, and so A', X', X, and C are collinear. Hence, $A'C = AX+BX+CX$, and we can apply the Law of Cosines to $\triangle A'BC$ to get that $A'C^2 = c^2+a^2-2ac\cos(B+60^\circ) = a^2+c^2+2ac\sin 60^\circ \sin B - 2ac\cos 60^\circ \cos B = a^2+c^2+2S\sqrt{3} - \frac{1}{2}(a^2+c^2-b^2) = \frac{a^2+b^2+c^2}{2} + 2S\sqrt{3} \implies PQ = \sqrt{\frac{2}{3}(a^2+b^2+c^2+4\sqrt{3}S)}$ (where S is again the area of ABC). Plugging in our values for a, b, and c, and using Heron's formula to find $S = \sqrt{10*5*3*2} = 10\sqrt{3}$, we can calculate $PQ = 2\sqrt{43}$.