1. **Answer:** $\frac{\mathbf{12}}{\mathbf{17}}$

   There are 21 ways out of 36 to roll two dice that sum up to a number greater than 6. The probability that the mathematician wins on his first turn is $21/36 = 7/12$. For him to wins in the second round, both he and the physicist must also lose in the first round, so the probability is $\left(\frac{5}{12}\right)^2 \frac{7}{12}$, and so on. Thus, the total probability is

$$\begin{aligned}\frac{7}{12} + \left(\frac{5}{12}\right)^2 \frac{7}{12} + \left(\frac{5}{12}\right)^4 \frac{7}{12} + \dots &= \left(1 + \frac{25}{144} + \left(\frac{25}{144}\right)^2 + \dots\right) \times \frac{7}{12} \\ &= \frac{144}{119} \times \frac{7}{12} = \frac{12}{17}\end{aligned}$$

2. **Answer:** $\frac{\mathbf{3}}{\mathbf{8}}$

   Assume without loss of generality that the first person gets a correct nametag. Let's call the other people B, C, D, and E. We can order the four people in nine ways such that none of the persons gets his own nametag; CBED, CDEB, CEBD, DBEC, DEBC, DECB, EBCD, EDBC, EDCB. Therefore, the desired probability is $\frac{9}{4!} = \frac{3}{8}$.

   Alternative Solution: The selection of random nametags amounts to a selection of a random permutation of the five students from the symmetric group S_5. The condition will be met if and only if the selected permutation σ has exactly one cycle of length one (i.e., exactly one fixed point). The only distinct cycle types with exactly one fixed point are $(1, 4)$ and $(1, 2, 2)$. There are $\dfrac{5!}{4} = 30$ permutations of the first type and $\dfrac{5!}{2^3} = 15$ permutations of the second. Thus, the desired probability is $\dfrac{30 + 15}{5!} = \dfrac{3}{8}$.

3. **Answer:** $\frac{\mathbf{49}}{\mathbf{729}}$

   Let the cube be oriented so that one ant starts at the origin and the other at $(1, 1, 1)$. Let x, y, z be moves away from the origin and x', y', z' be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves a, a', b, b' such that a precedes a' and b precedes b'. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in a, a', a, a' (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18 + 3 = 21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

4. **Answer:** $\mathbf{1006^2 = 1012036}$

   First note that the expression $(x + y + z)^n$ is equal to

$$\sum \frac{n!}{a!b!c!} x^a y^b z^c$$

   where the sum is taken over all non-negative integers a, b, and c with $a + b + c = n$. The number of non-negative integer solutions to $a + b + c = n$ is $\binom{n+2}{2}$, so $T_k = \binom{k+2}{2}$ for $k \geq 0$. It is easy to see that $T_k = 1 + 2 + \dots + (k + 1)$, so T_k is the $(k + 1)$st triangular number. If $k = 2n - 1$ is odd, then for all positive integers i, $T_{2i} - T_{2i-1} = 2i + 1$ and therefore [1]

$$\begin{aligned}\sum_{j=0}^{k-1} (-1)^j T_j &= T_0 + \sum_{j=1}^{n-1} (T_{2j} - T_{2j-1}) \\ &= 1 + \sum_{j=2}^{n} (2j - 1) \\ &= n^2.\end{aligned}$$

[1] For a quick visual proof of this fact, we refer the reader to `http://www.jstor.org/stable/2690575`.

Therefore, since T_{2010} is the 2011th triangular number and $2011 = 2(1006) - 1$, we can conclude that the desired sum is 1006^2.

5. **Answer: 44**

The minimum can be obtained by

$$1 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 3 + 3 \cdot 4 \cdot 1 + 4 \cdot 1 \cdot 2 = 12 + 12 + 12 + 8 = 44.$$

We claim that 44 is optimum. Denote $x_i = a_i b_i c_i$. Since $x_1 x_2 x_3 x_4 = (1 \cdot 2 \cdot 3 \cdot 4)^3 = 2^9 \cdot 3^3$, x_i should only consist of prime factors of 2 and 3. So between 8 and 12 x_i can only be 9.

Case 1. There are no 9 among x_i Then x_i are not in $(8, 12)$. And $x_1 x_2 x_3 x_4 = 12 \cdot 12 \cdot 12 \cdot 8$, so if x_1 is minimum then $x_1 \leq 8$. Then by AM-GM inequality $x_2 + x_3 + x_4 \geq 3(x_2 x_3 x_4)^{1/3}$. If we let $(x_2 x_3 x_4)^{1/3} = 12y$ then $x_1 = 8y^{-3}$, and for $y \geq 1$ $8y^{-3} + 36y$ attains minimum at $y = 1$. So $x_1 + x_2 + x_3 + x_4 \geq 8y^{-3} + 36y \geq 44$.

Case 2. x_1 is 9. Then $x_2 x_3 x_4$ is divisible by 3 but not 9. So only x_2 is divisible by 3 and others are just powers of 2. x_2 can be 3, 6, 12, 24 or larger than 44.

Case 2-1 $x_2 = 3 : x_3 x_4 = 2^9$, $x_3 + x_4 \geq 2^5 + 2^4 = 48 > 44$.

Case 2-2 $x_2 = 6 : x_3 x_4 = 2^8$, $x_3 + x_4 \geq 2^4 + 2^4 = 32$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 6 + 32 = 47$.

Case 2-3 $x_2 = 12 : x_3 x_4 = 2^7$, $x_3 + x_4 \geq 2^4 + 2^3 = 24$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 12 + 24 = 45$.

Case 2-4 $x_2 = 24 : x_3 x_4 = 2^6$, $x_3 + x_4 \geq 2^3 + 2^3 = 16$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 24 + 16 = 49$.

6. **Answer: $\mathbf{\frac{2011}{3}}$**

Let $n = 2011$ and $p = \frac{1}{3}$. The answer can be computed as follows

$$\begin{aligned}
\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} &= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} n \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} p^k (1-p)^{n-k} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= n \sum_{i=0}^{n-1} \binom{n-1}{i} p p^i (1-p)^{n-1-i} \\
&= np(p + 1 - p)^{n-1} \\
&= np
\end{aligned}$$

7. **Answer: 72381**

Observe that if the equation $ax + by = n$ has m solutions, the equation $ax + by = n + ab$ has $m + 1$ solutions. Also note that $ax + by = ax_0 + by_0$ for $0 \leq x_0 < b$, $0 \leq y_0 < a$ has no other solution than $(x, y) = (x_0, y_0)$. (It is easy to prove both if you consider the fact that the general solution has form $(x' + bk, y' - ak)$.) So there are ab such n and their sum is

$$\sum_{\substack{0 \leq x < b \\ 0 \leq y < a}} (ax + by + 2010ab) = 2010a^2b^2 + \frac{ab(2ab - a - b)}{2}.$$

Plug in $a = 2$ and $b = 3$ to arrive at the answer.

8. **Answer:** $\frac{\mathbf{1+(1/3)^{50}}}{\mathbf{2}}$

   The coin can turn up heads 0,2,4,..., or 50 times to satisfy the problem. Hence the probability is

$$P = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \cdots + \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0.$$

   Note that this sum is the sum of the even-powered terms of the expansion $(1/3+2/3)^{50}$. To isolate these terms, we note that the odd-powered terms of $(1/3-2/3)^{50}$ are negative. So by adding $(1/3+2/3)^{50}+(1/3-2/3)^{50}$, we get rid of the odd-powered terms and we are left with two times the sum of the even terms. Hence the probability is

$$P = \frac{(1/3+2/3)^{50}+(1/3-2/3)^{50}}{2} = \frac{1+(1/3)^{50}}{2}.$$

9. **Answer: 756**

   For any such function f, let $A = \{n \mid f(n) = n\}$ be the set of elements fixed by f and let $B = \{n \mid f(n) \in A \text{ and } n \notin A\}$ be the set of elements that are sent to an element in A, but are not themselves in A. Finally, let $C = \{1,2,3,4,5\} \setminus (A \cup B)$ be everything else. Note that any possible value of $f(f(x))$ is in A so A is not empty. We will now proceed by considering all possible sizes of A.

   (a) A has one element: Without loss of generality, let $f(1) = 1$, so we will multiply our result by 5 at the end to account for the other possible values. Suppose that B has n elements so C has the remaining $4-n$ elements. Since $f(f(x)) = 1$ for each x so any element c in C must satisfy $f(c) = b$ for some b in B, because $f(c) \neq 1$ and the only other numbers for which $f(x) = 1$ are the elements of B. This also implies that B is not empty. Conversely, any function satisfying $f(c) = b$ works, so the total number of functions in this case is $5\sum_{n=1}^{4}\binom{4}{n}n^{4-n}$ because there are $\binom{4}{n}$ ways to choose the elements in B, and each of the $4-n$ elements in C can be sent to any element of B (there are n of them). This sum is equal to $5(4+6\cdot 4+4\cdot 3+1) = 205$, so there are 205 functions in this case that A has one element.

   (b) A has two elements: This is similar to the first case, except that each element in B can now correspond to one of two possible elements in A, so this adds a factor of 2^n. The sum now becomes $\binom{5}{2}\sum_{n=1}^{3}\binom{3}{n}2^n n^{3-n} = 10(3\cdot 2+3\cdot 4\cdot 2+8) = 380$, so there are 380 functions in this case.

   (c) A has three elements: This is again similar to the prior cases, except there are 3 possible targes in A, adding a factor of 3^n. Then the sum is $\binom{5}{3}\sum_{n=1}^{2}\binom{2}{n}3^n n^{2-n} = 10(2\cdot 3+9) = 150$, so there are 150 functions in this case.

   (d) A has four elements: The logic is the same as the prior cases and there are $5(4) = 20$ functions in this case.

   (e) A has five elements: The identity function is the only possible function in this case.

   Adding together the five cases, we see that there are $205+380+150+20+1 = 756$ such functions.

10. **Answer: 23409**

    Let a_n be the number of ways of filling the $2\times 2\times n$ box, and let b_n be the number of ways of filling it with one $1\times 1\times 2$ box fixed at the "bottom face" (2×2 face). It is easy to see that $b_n = a_{n-1}+b_{n-1}$. It is then simple to verify that $a_n = 2b_n + 2b_{n-1} + a_{n-2}$. The base cases $a_1 = 2$, $b_1 = 1$, $a_2 = 9$, and $b_2 = 3$ are trivial to calculate. Using these values to calculate a_8 recursively gives $a_8 = 23409$.