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with thanks to Stephanie DeWet and Knots Knots by Justin Roberts

## Background and Definitions

What we usually think of as a knot is a 3-dimensional object. However, since we are unable to provide 3-dimensional paper, we will use a 2-dimensional definition. A *knot* is a single loop, possibly crossing over itself, with gaps left whenever it crosses to indicate one arc “above” and one arc “below”. Only two arcs may cross at a given point. (You can visualize this as a projection of a knot onto a sheet of paper.) Figure 1 shows two examples of knots.



Figure 1: Knots

Since our knots are two-dimensional, we must have rules for manipulating them. A convenient set of such rules is called the *Reidemeister moves*, or the *R-moves*, which act on some small portion of the knot, leaving the rest unchanged. (These should make sense if you again think of the knot as a projection. In fact, Reidemeister’s Theorem states that the R-moves represent all possible manipulations of a 3D knot that don’t involve cutting and rejoining it.) Figure 2 shows the four R-moves.

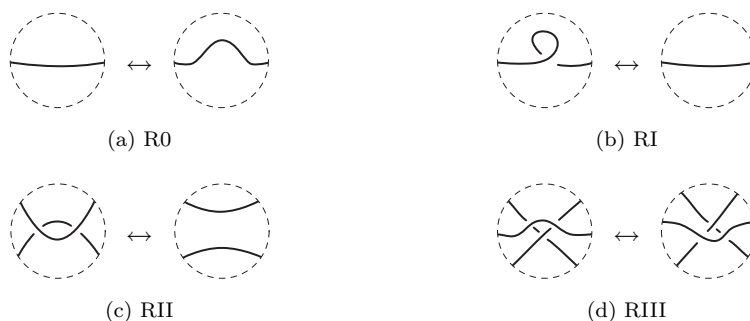


Figure 2: Reidemeister moves

Note that the R0 move is one we would take for granted; it allows deforming knots as desired as long as it doesn’t affect crossings. You also need not explicitly mention when you use R0, and you may combine more than one move in a single step, but be careful not to create confusing transformations by manipulating a knot with multiple moves without drawing intermediate steps. Note also that although these moves still apply to rotated versions of the depicted disks, they only apply with the crossings configured as shown (see problem 1).

We can without much extra trouble also consider links, which are like knots except composed of multiple loops. We call each loop a *component*; a knot is thus also a link (with one component). Link equivalence is defined just as knot equivalence. Figure 3 shows a few examples of links.



Figure 3: Links

## Instructions

Write/draw all solutions neatly, with at most one question per page, clearly numbered. Turn in the solutions in numerical order, with your team name at the upper right of every page. Be sure not to turn in two copies of a question, and note that if we can't decipher your drawings or handwriting, we can't grade it. Partial credit will be awarded where appropriate, and questions are weighted (though they are arranged in logical order, not order of difficulty).

Finally, remember that for  $i < j$  you may use the results of problem  $i$  on problem  $j$ , even if you have not solved problem  $i$ . For example, you may use the reverse R-moves of problem 1 in the rest of the problems.

## Questions

1. Show that the R-moves imply the first two (RI<sub>2</sub> and RIII<sub>2</sub>) *reverse R-moves* shown in Figure 4. You may use the remainder of the reverse R-moves without proof for later problems.

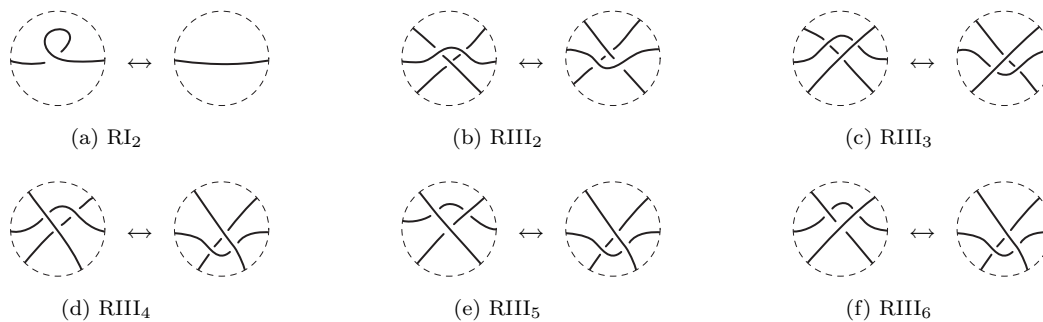


Figure 4: The reverse R-moves

2. Show that all knots with one or two crossings are equivalent to the *unknot*, a loop with no crossings, and is denoted  $U$ .
3. Show that the following two knots are equivalent.



A *knot invariant* is a function which depends only on equivalence class of knots, not on which particular representation is used. That is, if two knots are equivalent, the invariants of those knots are the same. (Note that the invariant might also be the same for two non-equivalent knots!) *Link invariants* are defined similarly.

4. (a) Let the *crossing number*  $c(K)$  be the minimal number of crossings of a knot  $K$  over all possible manipulations. Show that crossing number is a knot invariant.  
 (b) Show that the number of components  $\mu(L)$  of a link  $L$  is a link invariant.

An *oriented knot* is one with a chosen direction along the string, followed through the entire knot. An *oriented link* is defined similarly (note that a direction is needed for each component). We define *oriented knot invariants* and *oriented link invariants* as with unoriented knots and links. Note that it is sometimes helpful to draw multiple arrows on a single component to keep track of the orientation, and also that properties of unoriented knots generally apply to oriented versions as well. The R-moves and reverse R-moves work on oriented knots and links as well, preserving the directions of arrows. See figure 5 for examples of oriented knots and links. The R-moves apply to oriented knots and links as well, as long as you are careful to preserve the orientation of the segments as you move them.

Let  $D$  be an oriented link. The *linking number*  $Lk(D)$  is the sum over all crossings of contributions given by:

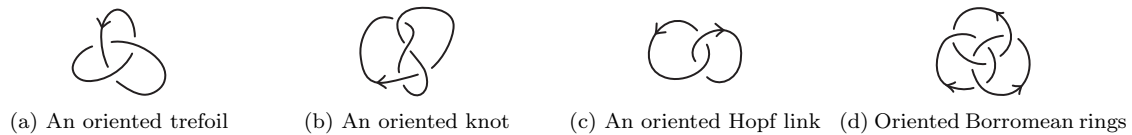


Figure 5: Oriented knots and links

- 0 if the arcs of the crossing belong to the same component
- $\pm 1$  if the arcs of the crossing belong to different components (see figure 6). We call the  $\pm 1$  the *sign* of the crossings.



Figure 6: The signs of crossings

- (a) Compute the linking number of the oriented Hopf link and Borromean rings shown in figures 5c and 5d.
- (b) Show that linking number is a knot invariant.
- (c) Show that linking number is a link invariant.

Suppose we color a link  $D$ , using a single color for each connected arc, such that at every crossing the three incident arcs (considering the arc on the bottom as two arcs instead of one) are either all the same color or all different colors. For example, the three connected arcs of the left trefoil knot (figure 1b) cannot be colored with two colors, since this would lead to two of one color and one of another at a crossing, but it can be colored with a different color for each arc. This is called a *3-coloring*, and  $\tau(D)$  denotes the number of possible 3-colorings of the link  $D$ .

- Calculate  $\tau$  for the trefoil knot (figure 1b).
- Show that  $\tau(D)$  is a link invariant.

The connect sum  $K_1 \# K_2$  of two oriented knots  $K_1$  and  $K_2$  is formed by cutting each in a single place, and joining them together so that the orientations of each knot match up, forming a single new oriented knot. Figure 7 shows an example of a connect sum. If the knots are unoriented, we form the connect sum without worrying about matching up orientations.

- Find, with proof, a formula for  $\tau(K_1 \# K_2)$  in terms of  $\tau(K_1)$  and  $\tau(K_2)$ .

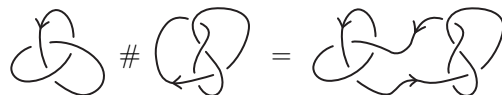


Figure 7: A connect sum

- Show that there are infinitely many distinct knots.

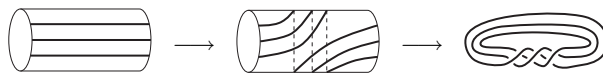


Figure 8: Construction of a torus knot

A torus link  $T_{p,q}$  is created by placing  $p$  string segments from end to end of a cylinder, evenly spaced around the circumference, twisting the cylinder through  $q/p$  full twists, and gluing the ends of the cylinder together, forming a torus out of the cylinder which is then removed, leaving only the string. The example in Figure 8 shows  $T_{3,3}$ , with all three segments moved to the front of the torus to make them visible.

10. Determine with proof the number of components of the link  $T_{p,q}$ . In particular, when is it a knot?

Another useful concept is the *Kauffman bracket* of an unoriented link  $D$ , denoted  $\langle D \rangle$ . It is a polynomial in integer powers of  $A$  defined by the following:

- It is invariant under R0 moves.
- It satisfies  $\langle D \rangle = A \langle D_+ \rangle + A^{-1} \langle D_- \rangle$ , where  $D_+$  and  $D_-$  are created by replacing a crossing as shown below, while leaving the rest of the knot unchanged.

$$\left\langle \begin{array}{c} \text{---} \\ \diagdown \\ \diagup \\ \text{---} \end{array} \right\rangle = A \left\langle \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\rangle$$

- It satisfies  $\langle D \amalg U \rangle = (-A^2 - A^{-2}) \langle D \rangle$ , where  $\amalg$  represents a disjoint sum. (A *disjoint sum* of two links is made by placing them together without entangling them and considering them one link. For example, the unlink of figure 3a is the disjoint sum of two unknots.) That is, removing a disjoint (unattached) unknot from the link results in a new link with Kauffman bracket  $(-A^2 - A^{-2})$  times that of the original link - see below for an example.

$$\left\langle \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\rangle = (-A^2 - A^{-2}) \left\langle \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\rangle$$

- $\langle U \rangle = 1$ ; the bracket of a crossingless unknot is 1.

11. Calculate the Kauffman bracket of the Hopf link and left trefoil knot, shown in figures 3b and 1b.

12. Determine which, if any, of the R-moves the Kauffman bracket is invariant under.

If  $D$  is an oriented link, then the *writhe*  $w(D)$  is the sum of the signs of all crossings of  $D$ . That is, it is the same as the linking number with self-crossings included as  $\pm 1$ , instead of being counted as 0.

13. Determine which, if any, of the R-moves the writhe is invariant under.

14. Show that the Jones polynomial  $f_D(A) = (-A^3)^{-w(D)} \langle D \rangle$  is an invariant of oriented links.

15. Find a formula for  $f_{D_1 \amalg D_2}(A)$  in terms of  $f_{D_1}(A)$  and  $f_{D_2}(A)$ , that is, for the Jones polynomial of a disjoint union in terms of the Jones polynomials of the two parts of the union.

16. Find a formula for  $f_{D_1 \# D_2}(A)$  in terms of  $f_{D_1}(A)$  and  $f_{D_2}(A)$ , that is, for the Jones polynomial of a connect sum in terms of the Jones polynomials of the two summands.

Reflecting a 3D knot across a plane in three dimensions causes all of the crossings in a corresponding (2D) knot to flip (the arc that was on top goes to the bottom, and vice versa). A knot is called *amphichiral* if it is equivalent to its mirror image, and *chiral* if it is not.

17. Find a method for determining whether a (2D) knot is amphichiral or chiral. Is the trefoil knot chiral?