Team Solutions
2007 Rice Math Tournament
February 24, 2007

1. **Answer: 2**

   By the Rational Roots Theorem, the roots must be ± 1 or $\pm p$. Check all of these; only -1 and p work.

2. **Answer: $\frac{1}{2}$**

   In the ab plane this is the square with vertices $(1,0),(0,1),(-1,0),(0,-1)$, which has area 2, while the total range of possible values is a 2×2 square.

3. **Answer: 10:34**

   Each angle between consecutive numbers on a standard clock is $30°$. So each minute, the hour hand moves $30/60 = (1/2)°$, and the minute hand moves $30/5 = 6°$. Letting the 12 o'clock position be 0, at 10:10, the small angle difference between the two hands is $360-(300+1/2\cdot 10)+60 = 115$. In m minutes, the angle between the hand positions will be $115+6m-\frac{1}{2}m$. The angle will actually be the same when this is $360-115 = 245$; solving for m gives Set those equal, and solve for $m = 260/11 = 23+7/11$; adding back up gives the answer.

4. **Answer: 6**

   The triangle must have at least one side not in the direction of axes, and since $\{3,4,5\}$ is the smallest Pythagorean triplet, the smallest length for this side is 5, and we can minimize the area by making the remaining two sides 3 and 4.

5. **Answer: 123**

   Look at how many five-letter non-words you can make. Let C be a consonant and let V be a vowel. The possible patterns are: $CCCVC$, $CVCCC$, $CCCCV$, $VCCCC$ (16 words each); $CCCVV$, $VCCCV$, $VVCCC$ (8 words each); $CCCCC$ (32 words). Thus there are 120 non-words. There are $3^5 = 243$ sequences, so there are $243-120 = 123$ words.

6. **Answer: 1044**

   First consider $y \equiv \left(\sum_{k=1}^{2015} k\right) \mod 2016$: the sum of the first and last term, of the second and next to last, and so on is always 2016. However, there will be one term left in the middle, 1008. Thus $y \equiv 1008 \mod 2016$. We have added in $2015+2014+\cdots+2008 \equiv (-1)+(-2)+\cdots+(-8) \equiv -36 \mod 2016$, so finally $x \equiv 1008-(-36) \equiv 1044 \mod 2016$.

7. **Answer: 645**

   Daniel: There are $\frac{5!}{2!}$ integers using all the digits. Dropping one, in 3 cases (like 1,2,2,3) there are $\frac{4!}{2!}$ integers and (dropping a 2) there are 4!. Dropping 2, there 3! integers in 4 cases (like 1,2,3) and $\frac{3!}{2!}$ in three cases (like 1,2,2). Dropping 3, there are 2! integers in 4 cases (like 1,2) and also 22. Finally, there are 4 one-digit numbers. Adding up gives 170. For Edward, there are $\frac{6!}{2!3!}$ words if he drops a B or S, $\frac{6!}{3!}$ if he drops an N, and $\frac{6!}{2!2!}$ if he drops an A, totalling to 420. Finally, Fernando can distribute candy by putting the nine pieces in a line and drawing two lines to divide it into three piles. There are $\binom{10}{2}$ ways to do this without drawing the two lines in the same place, and another 10 with the two lines in the same place.

8. **Answer: 120π**

   First we have that the distance to the horizon is $\sqrt{(13+156)^2-156^2} = 13\sqrt{13^2-12^2} = 65$. Let r be the length of an altitude to hypotenuse of 65-156-169 triangle; this is the radius of the ant's circular path. By similar triangles we have $r = \frac{65\cdot 156}{169} = \frac{5\cdot 13}{13}\cdot\frac{12\cdot 13}{13} = 5\cdot 12$. Path length $= 2\pi r = 120\pi$.

9. **Answer: $\boldsymbol{2n-2}$**

   To write a derangement of length n, we first pick i from among the $n-1$ possibilities for the first element. If we place 1 in the i^{th} position, there are d_{n-2} ways left to derange the remaining $n-2$ integers, since their positions are all left. There are d_{n-1} ways to place 1 somewhere other than the i^{th} position and derange the remaining, since we might as well rename 1 to i in this case. Thus the recurrence relation is $d_n = (n-1)(d_{n-1} + d_{n-2})$.

10. **Answer: $\boldsymbol{\frac{l}{\pi}}$**

    For any polygon with an inscribed circle of radius r, the area is $A = \frac{1}{2}rP$, where P is the perimeter. This is true since we can cut the polygon up into right triangles, each with one leg of length r and the other leg a part of the perimeter; adding up the areas of all the triangles and factoring out the latter lengths we obtain $A = \frac{1}{2}rP$. Thus $A = \frac{1}{2}r \cdot 4l = 2\pi r^2 \Rightarrow rl = \pi r^2 \Rightarrow r = \frac{l}{\pi}$.

11. **Answer: 2005**

    The degree of R cannot be greater than the degree of the denominator polynomial, so we can write $R(x) = ax + b$. $x^{2007} = Q(x) \cdot (x^2 - 5x + 6) + rx + s$, where Q is some polynomial. Set $x = 2$ and $x = 3$ successively to get $x^{2006} = 2r + s$ and $3^{2006} = 3r + s$. Solving for s, we get $R(0) = s = 2 \cdot 3 \cdot (2^{2006} - 3^{2006})$.

12. **Answer: $\boldsymbol{\min\{t, s\}}$**

    Suppose without loss of generality $s < t$. Then by independence of past and future, the average of $(B(t) - B(s))^2 = t - s$. But $(B(t) - B(s))^2 = B(t)^2 - 2B(t)\overline{B(s) + B}(s)^2$. Averaging both sides and plugging in for the averages of squares, we have $t - s = t - 2\overline{B(t)B(s)} + s$, so the desired average is s. This is of course true if $t = s$ as well.

13. **Answer: $\boldsymbol{\frac{91}{29}}$**

    Let $p = \frac{7}{8}$ and $q = \frac{5}{8}$. For the volley to end after an odd number of returns, the probability is $pqpq \cdots pqp(1-q) = (pq)^{(n-1)/2}p(1-q)$, and for it to end after an even number of returns it is $pqpq \cdots pq(1-p) = (pq)^{n/2}(1-p)$. Writing $n = 2k+1$ and $n = 2k$ for odd and even, the desired average is then $\sum_{k=0}^{\infty} 2k(1-p)(pq)^k + \sum_{k=0}^{\infty}(2k+1)p(1-q)(pq)^k$. To evaluate these, we try:

    $$\begin{aligned}\sum_{n=0}^{\infty} np^{n-1} &= 1 + 2p + 3p^2 + \cdots \\ &= (1 + p + p^2 + \cdots) + (p + p^2 + \cdots) + (p^2 + \cdots) + \cdots \\ &= (1 + p + p^2 + \cdots)(1 + p + p^2 + \cdots) = \frac{1}{(1-p)^2}\end{aligned}$$

    Applying this, multiplying back in the other constants, and simplifying gives $\frac{p(1+q)}{1-pq}$, and then we just plug in for p, q.

    Answer: $\boldsymbol{pn^2 + qn}$

    First we solve the recurrence relation for x_n, obtaining $x_n = x_{n+1} + p - \sqrt{q^2 + 4px_{n+1}}$. We rewrite this as $x_{n-1} = x_n + p - \sqrt{q^2 + 4px_n}$ and add to the original recurrence relation to get $x_n + 1 - 2x_n + x_{n-1} = 2p$. Let $x'_n = x_{n+1} - x_n$. Thus $x'_n - x'_{n-1} = 2p$, and since $x'_0 = x_1 - x_0 = p + q$, $x'_n = 2pn + (p+q)$. Since the first difference of the sequence is linear, the next is constant and so we have a quadratic. The second difference $2p$ is twice the coefficient of n^2, and considering x_0 and x_1 yields the coefficient of n and the constant, giving the desired solution.

14. **Answer: $\boldsymbol{\frac{\pi}{2} \log \pi}$**

    Let $I(a) = \int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1} x}{x} dx$. Then $\frac{dI}{da} = \int_0^{\infty} \frac{1}{1+(ax)^2} dx = \frac{1}{a} \tan^{-1} x\Big|_0^{\infty} = \frac{\pi}{2a}$. $I(a) = \int \frac{dI}{da} da = \frac{\pi}{2} \log a + C$, but clearly $I(1) = 0$ so $C = 0$.