General Solutions
2007 Rice Math Tournament
February 24, 2007

1. **Answer: 1**

   Pick up a piece of fruit in the "mixed" bin; say it is an apple. Then you know that this bin should be labeled "apples," and the bin currently labeled "apples" must be "oranges" (since you know that the oranges label was incorrect). And so the current "oranges" should be "mixed."

2. **Answer: 97**

   If there are x Lumixians and y Obscrans, $x + y = 60$ and $4x + y = 129$; solving gives $x = 23$ and $y = 37$, so $x + 27 = 97$.

3. **Answer: $\frac{1}{3}$**

   Since there are three red marbles, drawing a red marble means that 2/3 of the time she has drawn from the box with two red marbles, so only 1/3 of the time is the other marble blue.

4. **Answer: 1**

   Note that $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$; writing out the tangents as sines divided by cosines, we see that $1 = \tan 10° \tan 80° = \tan 20° \tan 70° = \cdots$.

5. **Answer: $\frac{4\pi}{3}$**

   The arc forming half the boundary of the intersection is of measure $120°$, so the circumference is $2 \cdot \frac{120}{360} \cdot 2\pi$.

6. **Answer: $\frac{65}{81}$**

   $P(\text{lose}) = 1 - P(\text{lose all 5}) = 1 - (\frac{2}{3})^4 = \frac{65}{81}$.

7. **Answer: $\frac{5}{2}$ or 2.5**

   Essentially, the boat is traveling at 4 mph toward a stationary branch 10 miles away. It will take 10/4=2.5 hours to reach it.

8. **Answer: $\frac{3}{8}$**

   Let Silas receive a letter from Jessica: then there are 3 ways to arrange the remaining envelopes and letters by letting Jessica receive letters from either Silas, Katie, or Lekan. The same is true if Silas receives the letter from Katie or Lekan, so there are $3 \cdot 3 = 9$ total ways for no one to receive the correct letter. There are a total of 4! ways to receive letters, so the probability we want is $9/4! = 3/8$.

9. **Answer: 10**

   After n days, Peter will have earned $6.4n$ dollars, while Crocodile will have earned $\frac{0.1(2^n-1)}{2-1}$ dollars. We want the n to satisfy $6.4n < \frac{0.1(2^n-1)}{2-1}$. Simplifying: $n < 2^{n-6} - \frac{1}{64}$. Since $9 > 2^{9-6} = 8$ but $10 < 2^{10-6} = 16$, so the smallest such n is 10.

10. **Answer: 18**

    Draw a Venn diagram - we can put the most in the center if none are friends with just two of them. Let x be the number of students friends with all 3; then $24 - x$, $20 - x$, and $39 - x$ are friends with just Al, Betty, and Clara respectively. Adding up, $x + (24 - x) + (20 - x) + (39 - x) = 47$, since they are 3 members of the class of 50, so $x = 18$.

11. **Answer: 23520**

    There are 8! arrangements of the letters, but we divide by 3! and 2! since switching the a's or the r's has no effect. There are then seven ways to split each of these into a first and last name, giving $8! \cdot 7/(2!3!) = 8 \cdot 7 \cdot 7 \cdot 5 \cdot 4 \cdot 3$ possible names.

12. **Answer: $\boldsymbol{5\pi}$**

    After nine slices, he has cut $9 \cdot 30° + 0° + 2° + \cdots + 16° = 270° + 18° \cdot 4 = 342°$, so the leftover slice is $18°$, with area $100\pi \cdot 18°/360° = 5\pi$.

13. **Answer: 2**

    The ratio of distances along the rope, ground (shadow), and above the ground is $10 : 6 : 8$, so having travelled $30 \cdot 5\ cm = 1.5m$ along the ground, it has travelled $8 \cdot 1.5/6\ m = 2\ m$ above the ground.

14. **Answer: 148**

    $$\sum_{j=1}^{49} (a_{i+1} - a_i) = a_{50} - a_1 \leq 149 - 1$$

15. **Answer: $\mathbf{\frac{3}{4}}$**

    We draw the rectangle from 0 to 1 on the x-axis and -1 to 1 on the y-axis; $x > y$ is the region below a line from the origin to $(1, 1)$. This region has area $\frac{3}{2}$ out of the total area of 2.

16. **Answer: 3**

    Let s be the side length of the square. Looking at one of the 30-60-90 triangles outside the square would give $\frac{2+\sqrt{3}-s}{2} \cdot \sqrt{3} = s$, so $s = \sqrt{3}$.

17. **Answer: 50%**

    Let $T^{+/-}$ indicate the test result and $B^{+/-}$ indicate whether the person actually does or does not have bifurcation virus. The probability that someone has the virus, given that their test is positive, is equal to the probability that a given person tests positive and has it over the total probability of testing positive. In statistical notation:

    $$\begin{aligned} P(B^+|T^+) &= \frac{P(T^+|B^+) \cdot P(B^+)}{P(T^+|B^+) \cdot P(B^+) + P(T^+|B^-) \cdot P(B^-)} \\ &= \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + (1 - 0.99) \cdot (1 - 0.01)} \\ &= 1/2. \end{aligned}$$

18. **Answer: 900**

    Since $L = 1200 - 2W$, Area$=W \cdot L = W(1200 - 2W) = -2W^2 + 1200W$. This is a parabola, so its vertex is at $W = 1200/(2 \cdots 2) = 300$, giving $L = 600$.

19. **Answer: $\boldsymbol{a, d, c, b}$**

    We know $d > a$: $100! > 10^100$ since each is a product 100 numbers but the first has larger ones. We know $c > d$ since $1000000! > (100000!)^{10} > (100!)^10$. Finally, $b > c$ since $1000000! < 1000000^{1000000} = (10^6)^{10^6} = 10^{6 \cdot 10^6}$.

20. **Answer: $\mathbf{2n^2}$**

    There are a total of $\binom{4n+2}{2} = 8n^2 + 6n + 1$ pairs of paths; subtracting the pairs which represent intersections of only two lines, there are $6n^2$ pairs of paths left. Each point at which three paths intersect accounts for three pairs, so there are at most $\frac{6n^2}{3}$ points at which exactly three paths intersect.

21. **Answer:** $\mathbf{\frac{5111}{5555}}$

$$10^5x = 92007.\overline{2007} \text{ and } 10x = 9.\overline{2007}$$
$$10^5x - 10x = 92007.\overline{2007} - 9.\overline{2007} = 91998$$
$$x = \frac{91998}{99990} = \frac{5111}{5555}$$

22. **Answer:** $\mathbf{\frac{137}{6}}$

If an event happens with probablity p each second, the expected number of seconds for it to occur is $1/p$ (this is a definition of probability!). Thus adding up for each dropped ball, the expected time until no balls are left is $\frac{10}{5} + \frac{10}{4} + \frac{10}{3} + \frac{10}{2} + \frac{10}{1} = 10(\frac{12+15+20+30+60}{60}) = \frac{1}{6} \cdot 137 = \frac{137}{6}$ seconds.

23. **Answer: 11**

Note that in a quadrilateral $ABCD$ we have $AB + BC + CD > AD$, since $AB + BC > AC$ and $AC + CD > AD$ by the triangle inequality. This gives us three possible equations: $3 + x + y > 3$ (this will always be true), $x + 6 > y$, and $y + 6 > x \Rightarrow y > x - 6$. Thus $x + 6 > y > x - 6$, giving 11 possible values from $x - 5$ to $x + 5$.

24. **Answer:** $\mathbf{\frac{11}{32}}$

For Andy (A) to win, he must win 4 points before Bob wins 3.

$P(\text{A wins on 4th point}) = \left(\frac{1}{2}\right)^4$

$P(\text{A wins on 5th point}) = (\text{Ways to happen}) \cdot P(\text{A wins 4}) \cdot P(\text{A loses 1}) = 4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)$

$P(\text{A wins on 6th point}) = \binom{5}{2} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2$

Sum: $\left(\frac{1}{2}\right)^4 + 4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right) + \binom{5}{2} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{22}{64} = \frac{11}{32}$.

25. **Answer: 12**

Let M be the endpoint of the altitude on the hypotenuse. Since we are dealing with right triangles, $\Delta MAC \sim \Delta ABC$, so $AM = 12/5$. Let N be the endpoint he reaches on side $\overline{AC}$. $\Delta MAC \sim \Delta NAM$, so $\frac{MN}{AM} = 4/5$. This means that each altitude that he walks gets shorter by a factor of $4/5$. The total distance is thus $\frac{12}{5} / \left(1 - \frac{4}{5}\right) = 12$.