Advanced Topics Test
2006 Rice Math Tournament
February 25, 2006

1. A college student is about to break up with her boyfriend, a mathematics major who is apparently more interested in math than her. Frustrated, she cries, "You mathematicians have no soul! It's all numbers and equations! What is the root of your incompetence?!" Her boyfriend assumes she means the square root of himself, or the square root of i. What two answers should he give? Express your answer in the form $a + bi$. *(The answer form was added after the contest)*

2. Define $A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$. Find a vertical vector v such that $(A^8 + A^6 + A^4 + A^2 + I)v = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$ (where I is the 2×2 identity matrix).

3. Simplify: $\sum_{k=10}^{2006} \binom{k}{10}$ (Your answer should contain no summations but may still contain binomial coefficients/combinations).

4. Rice University and Stanford University write questions and corresponding solutions for a high school math tournament. The Rice group writes 10 questions every hour but make a mistake in calculating their solutions 10% of the time. The Stanford group writes 20 problems every hour and makes solution mistakes 20% of the time. Each school works for 10 hours and then sends all problems to Smartie to be checked. However, Smartie isn't really so smart, and only 75% of the problems she thinks are wrong are actually incorrect. Smartie thinks 20% of questions from Rice have incorrect solutions, and that 10% of questions from Stanford have incorrect solutions. This problem was definitely written and solved correctly. What is the probability that Smartie thinks its solution is wrong?

5. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2}+(k+2)\sqrt{k}}$

6. Ten teams of five runners each compete in a cross-country race. A runner finishing in n^{th} place contributes n points to his team, and there are no ties. The team with the lowest score wins. Assuming the first place team does not have the same score as any other team, how many winning scores are possible?

7. A lattice point in the plane is a point whose coordinates are both integers. Given a set of 100 distinct lattice points in the plane, find the smallest number of line segments $\overline{AB}$ for which A and B are distinct lattice points in this set and the midpoint of $\overline{AB}$ is also a lattice point (not necessarily in the set).

8. The following computation arose in the research of mathematician P.D.: Let $k_i(j) = \frac{n+1}{2n+1} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}}$ for $0 \le i, j \le n$.

$$\sum_{j=0}^{n} k_i(j).$$

9. How many positive integers appear in the list $\left\lfloor \frac{2006}{1} \right\rfloor, \left\lfloor \frac{2006}{2} \right\rfloor, \ldots, \left\lfloor \frac{2006}{2006} \right\rfloor$ where $\lfloor x \rfloor$ represents the greatest integer that does not exceed x?

10. Evaluate:

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 - n + 1}\right)$$