Power Test Solutions
2005 Rice Math Tournament
February 26, 2005

1. **Answer: $(-1.5, 2)$**
The distance from $(-6, 8)$ to the origin is 10, so the distance between its inversion and $(0, 0)$ is 2.5. Clearly, the inversion must lie on $y = -\frac{4}{3}x$ and $x < 0$. The answer is $(\mathbf{-1.5}, \mathbf{2})$.

2. **Answer: $(5.4, 9.2)$**
The circle is centered at $(4, 9)$ with $r^2 = 10$ and distance of $\sqrt{50}$ from $(11, 10)$, so the distance between the inversion and $(11, 10)$ is $\frac{10}{\sqrt{50}}$. The inversion must lie on $\frac{y-9}{x-4} = \frac{1}{7}$. The new point is shifted 1.4 in the $+x$ direction and 0.2 in the $+y$ direction to give $(\mathbf{5.4}, \mathbf{9.2})$.

3. **Answer: $\mathbf{p = x_0 + \frac{r^2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}, q = y_0 + \frac{r^2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}}$**

Call the new point (p, q). Set $y' = y - y_0$ and $x' = x - x_0$. Set $d^2 = y'^2 + x'^2$, and let s be the distance from (p, q) to (x_0, y_0). So $s = \frac{r^2}{d}$. Further define $p' = p - x_0$ and $q' = q - y_0$. It is clear from the definition of inversion that p' and x' must be of the same sign and likewise with q' and y'. By similar triangles, $\frac{p'}{x'} = \frac{q'}{y'} = \frac{s}{d}$. Thus $p' = \frac{x's}{d} = \frac{x'r^2}{d^2}$, and $q' = \frac{y'r^2}{d}$. It follows that $\mathbf{p = x_0 + \frac{r^2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}}$ and $\mathbf{q = y_0 + \frac{r^2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}}$.

4. a. All three criteria for inversion must be met. Since P' lies on the line defined by C and P, all three points are collinear, and P lies on the line containing P' and C. The second criterion is immediately satisfied, as C is not contained in the line segment PP', which is equivalent to $P'P$. The final criterion is valid because $(CP')(CP) = (CP)(CP') = r^2$.

   b. If $P = P'$, then $CP = CP'$. Since distances are positive, each must be r to satisfy the third criterion. Since $CP = r$, P lies on circle C.

5. By symmetry, it is clear that Y is equidistant from A and B and hence lies on line CP. It is also clear that C does not lie between P and Y. Clearly, angles $\angle CAY$, $\angle CBY$, $\angle CPA$, and $\angle CPB$ are all right angles. By reflexivity of angle $\angle ACP$, triangle $\triangle ACP$ is similar to triangle $\triangle YCA$ and so $\frac{CP}{CA} = \frac{CA}{CY}$, implying that $r^2 = (CP)(CY)$.

6. a. There are two cases. In the first, point C does not lie between A and B. The line segment AB is then a set of points a distance d from point C such that d is between distances CA and CB, inclusively. Each such point is projected to a point a distance $\frac{r^2}{d}$ from C and on the same side of C as A and B. Thus the inversion set is the set of all points a distance $\frac{r^2}{d}$ from C, on the same side of C as A and B, and collinear with A and B for all d between CA and CB inclusively. This is the **line segment $\mathbf{A'B'}$**, where A' and B' are inversions of A and B respectively. If C is between A and B, we will have to consider segments CA and CB separately. Segment CA is the set of collinear points a distance d from C, where d is positive and no larger than CA. This projects to a set of points collinear with CA with a distance at least CA' from C and on the same side of C as A. Likewise with segment CB. The result is the **set of points contiained in the line through $\mathbf{A'}$ and $\mathbf{B'}$ but not in the interior of the line segment $\mathbf{A'B'}$**.

   b. **Answer: a line through C is its own inversion**
The inversion set must be a subset of the line itself because of collinearity. Every point on the line a distance d from point C projects to a point a distance $\frac{r^2}{d}$ from C and on the same side of C as the original point. Since d takes on any real number value (and since point C itself cannot be inverted) and since points on both sides of C are considered, the **line through C is its own inversion**.

7. a. **Answer:** $(\mathbf{\frac{r^2}{d}, 0})$
The inversion lies on the positive *x-axis*. It is a distance $\frac{r^2}{d}$ from $(0,0)$, so it is at $(\mathbf{\frac{r^2}{d}, 0})$.

b. **Answer: approaches** $(\mathbf{0, 0})$
As y approaches infinity or negative infinity, the distance between (d, y) and $(0, 0)$ approaches infinity, so the distance between the inversion of (d, y) and $(0, 0)$ approaches zero. Hence the inversion of (d, y) **approaches** $(\mathbf{0, 0})$

c. Let $A = (\frac{r^2}{d}, 0)$, $B = (d, 0)$, and let D be a point on line L other than A. Then $(CA)(CB) = (CD')(CD)$ where C' is the inversion of D. Hence $\frac{CA}{CD} = \frac{CD'}{CB}$. Since angle $\angle DCB$ is reflexive, it follows that triangle $\triangle DCB$ is similar to triangle $\triangle ACD'$. Since angle $\angle DBC$ is a right angle, so is angle $\angle AD'C$. thus, D' traces out a circle with diameter AC. All points on this circle are included since CD' can take on any positive value no larger than AC. This is true since CD takes on all positive values no smaller than AC. Hence, the inversion of line L is a circle **centered at** $(\mathbf{\frac{r^2}{2d}, 0})$ **with radius** $\mathbf{\frac{r^2}{2d}}$. This problem can also be done analytically to yield the same result.

8. If a line intersects a given circle centered at C at two points, A and B, that are not diametrically opposed, the inversion of the line about the circle is the circle through C, A and B (from problem 7). Since the inversion of any point on the interionr of C lies on its exterior (and vice versa), the inversion of a chord contained in the line is the portion of the inversion circle that lies outside of circle C.

One way to solve this problem is to consider all lines parallel to line segment AB (one of which contains AB) that lie at least as far from point C as the line through points A and B and that intersect circle C at one or two points. The inversion of this set is clearly the set of points bounded by two circles: the circle through points C, A, and B, and the circle with diameter CM where M is the midpoint of minor arc AB. The inversion of the set in question is the portion of the above locus that lies outside of circle C. This is the **set of all points contained in the interior of the circle through points C, A, and B and outside of circle C, including boundaries**. This solution may be shown geometrically too.

9. a. Suppose the circle containing P has radius a and it centered at point O. Without loss of generality, we assume P is on the interior of circle C. Let K be the point on circle O that is collinear with C and P and not equal to P. Let T and U be the distinct points on circle O that lie on line CO with T between C and O. Orthogonality implies $(CO)^2 = r^2 + a^2$. Hence $(CO)^2 - a^2 = (CO - a)(CO + a) = (CT)(CU) = r^2$. Note that angles $\angle PKT$ and $\angle PUT$ are equal since they correspond to the same minor arc PT. Angle $\angle C$ is reflexive so triangles $\triangle CKT$ and $\triangle CUP$ are similar. Hence $\frac{CT}{CK} = \frac{CP}{CU}$ and $(CP)(CK) = (CT)(CU) = r^2$. Thus K is the inversion of P about circle C.

b. Again center the circle through P and P' at point O and denote its radius by a. Points T and U are defined as above. Since $(CP)(CP') = r^2$, the same argument as above (Secand-secant power theorem) can be used to show that $(CT)(CU) = r^2 = (CT)(CT + 2a) = (CT)^2 + 2a(CT)$. Adding a^2 to each side yields: $r^2 + a^2 = (CT + a)^2 = (CO)^2$. Hence, angle circles C and O intersect at a right angle and are orthogonal.

10. Circles C and D intersect at exactly 0, 1, 2, or infinitely many points. In the first case circle D is entirely contained in the interior or exterior of a circle C. Since the inversions of such sets must lie entirely in the exterior or interior of circle C, respectively, circle D cannot be its own inverse. In the second case, a similar argument applies - the only difference is that D and C are ttangent and only intersect at one point. If the circles intersect at all points, they are the same, and it is trivial to show the circle D is its own inverse. If the circles inversect at exactly 2 points, the part of circle D on the interior of circle C must project to the exterior part. Letting P be a point on circle D that is on the

interior of circle C, P' is on circle D and is collinear with points C and P. From problem 9b, we see that circles C and D are orthogonal.

11. a. Let P and Q be distinct points on circle K that are collinear with point C. Let S be a point on circle K so that CS is tangent to the circle. The secant-tangent power theorem shows that $(CP)(CQ) = (CS)^2 = w^2 = a^2$, and this can be easily seen by examining triangles $\triangle CPS$ and $\triangle CSQ$. It follows that $\frac{CP'}{CQ} = \frac{(CP')(CP)}{w^2-a^2} = \frac{r^2}{w^2-a^2}$. Thus CP' is a dilation of CQ by a constant factor, so the triangle $\triangle CQK$ is dilated by $CP'K'$ (where K' is the dilation of K by the above factor). This means that $P'K'$ is a constant factor of $QK = a$. Hence, $P'K'$ is constant, and the dilation is indeed a circle.

    b. **Answer:** $\mathbf{\frac{ar^2}{w^2-a^2}}$.
    If P and Q are diametrically opposed, then CP' and CQ' are equal to $\frac{r^2}{w-a}$ and $\frac{r^2}{w+a}$ in either order. The difference is $\frac{2ar^2}{w^2-a^2}$, making the radius equal to $\mathbf{\frac{ar^2}{w^2-a^2}}$.

12. First consider the angle between the tangent at P to C_1 and line OP. Let T be a point on C_1, and let T' and P' denote the inversion of T and P, respectively. Clearly, $(OP)(OP') = (OT)(OT')$, so $(OT')/(OP) = (OP')/(OT)$. By the reflexivity of angle $\angle POT$, triangles $\triangle PTO$ and $\triangle T'P'O'$ are similar. This suggests that angle $\angle OP'T'$ is congruent to the angle between OT and TP. As T approaches P, lines TP and $T'P'$ approach tangency at C_1 and C_1'. Hence angle $\angle OP'T'$ approaches the angle between OP and the tangent to C_1' at P', and the angle $\angle OTP$ approaches the angle between OP and the tangent to C_1 at P. The same argument can be made with curve C_2. The angles between the tangents at P and P' are split by OP to produce eequivalent smaller angles. Thus, the original angles made by the tangents are equal.

13. a. First, one must show that $(AP)(AQ)$=constant. Let V be the intersection between segments BC and AQ. Note that $(AV-PV)(AV+PV) = (AV)^2-(PV)^2 = (x^2-(BV)^2-(y^2-(BV)^2) = x^2-y^2$. Call this constant k^2. Then let Z be a point in the plane that is a distance of k from A and P. Thus the condition $PZ = AZ$ suggests that P is confined to a fixed circle, where A and Z are also fixed. Likewise, Q is the inversion of P about a hypothetical circle centered at A with radius k. Circle Z clearly contains A and is internally tangent to circle A. From problem 7, it follows that Q is confined to a line segment. This geometric constraints of the diagram prohibit P from traversing the entire circle, so Q is confined to a line segment, not a full line.

    b. **Answer:** $\mathbf{2y(x+y)}$
    The distance AQ varies between k (when $P = Q$) and $(x + y)$ at the extremes. This can be shown geometrically. The distance is calculated with the Pythagorean theorem to be $\mathbf{2y(x+y)}$.