Power Test
2005 Rice Math Tournament
February 26, 2005

Inversions of points

Consider a circle with center C and radius r. This circle can be referred to as "circle $\bigcirc C$". If P is a point other than C in the plane of the circle, then the inversion of P about circle $\bigcirc C$ is denoted P' and, by definition, satisfies the following rules:

(1) Point P' lies on the line through C and P
(2) Point C is *not* contained in the line segment PP' (Point C is not "between" P and P')
(3) The product $(CP)(CP') = r^2$

In this case, circle $\bigcirc C$ is called the *circle of inversion* and point C is the *center of inversion*. Assume for convenience that the inversion of point C about circle $\bigcirc C$ is undefined.

Justify all solutions for full credit.
Totalpoint value: 240 points

Note: For all problems, you may refer to correct results of earlier questions even if you do not prove those results correctly. Be sure to state the problem number when referencing a result (e.g.- "The results of problem 7c suggests ...") However, you may NOT refer to results of later questions to answer any problems (e.g. - the result of problem 3 cannot be used to answer problems 1 and 2).

1.) (5 pts.) Find the inversion of point $(-6, 8)$ about the circle centered at $(0, 0)$ with radius 5.

2.) (5 pts.) Find the inversion of point $(11, 10)$ about the circle $(x-4)^2 + (y-9)^2 = 10$

3.) (20 pts.) Generalize the previous results by finding the inversion of a point (x, y) about the circle centered at (x_0, y_0) with radius r.

4.) (8 pts. total)
a.) Suppose a circle with radius r is centered at point C. A point P not equal to C is in the plane, and the inversion of P about $\bigcirc C$ is P'. Prove that the inversion of P' about $\bigcirc C$ is P. (5 pts.)
b.) Show that if $P = P'$, then P is on circle $\bigcirc C$. (3 pts.)

5.) (15 pts.) A circle is centered at C and has radius r. A point P in the plane of the circle is a distance d from C, and $0 < d < r$. The line through P that is perpendicular to line CP intersects the circle at points A and B. The lines tangent to circle $\bigcirc C$ at A and B intersect at point Y. Show that Y is the inversion of P about circle $\bigcirc C$.

Inversions of Sets

Consider a circle centered at C, and suppose S is a set of points in the plane, none of which is C itself. The inversion of S about circle $\bigcirc C$ is the set S' of the inversions of all points in S. In other words, for every point P in set S, P' is an element of S'.

6.) (13 pts. total) A circle has center C and radius r. Points A and B in the plane of the circle are distinct from but are collinear with C (not in any particular order).
a.) Find the inversion of the line segment AB. (8 pts.)
b.) Find the inversion of a line through C with C removed. (5 pts.)

7.) (30 pts. total) A circle centered at $(0, 0)$ has radius r. Let L be the line defined by $x = d$ where $d > 0$ is a constant.
a.) What is the inversion of $(d, 0)$ about the circle? (5 pts.)
b.) What happens to the inversion of (d, y) about the circle as y approaches infinity and negative infinity? (5 pts.)

c.) Show that the inversion of L about the circle centered at the origin with radius r is a circle and determine its center and radius. (20 pts.)

8.) (20 pts.) Points A and B are chosen on a circle with center C, and the line segment AB is not a diameter of the circle. Describe or precisely show the inversion of the set of all points in the region bounded by line segment AB and the minor arc AB about circle $\bigcirc C$ (include the boundaries in the set).

Orthogonal Circles
Two circles $\bigcirc C$ and $\bigcirc D$ are said to be orthogonal if they intersect at two points and if at each point of intersection, the lines tangent to $\bigcirc C$ and $\bigcirc D$ at that point are perpendicular.

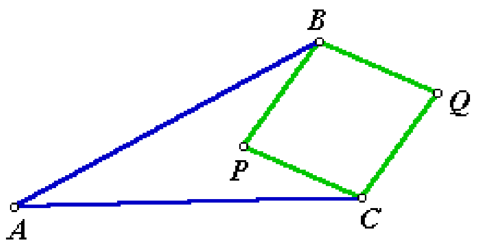
9.) (36 pts. total) Consider a circle with center C and radius r. A point P not equal to C and its inversion P' lie in the plane of the circle.
a.) Show that if a given circle through P is orthogonal to circle $\bigcirc C$, then it must also contain P'. (18 pts.)
b.) Show that if a given circle through P also contains P', then it is orthogonal to circle $\bigcirc C$. (18 pts.)

10.) (10 pts.) Two circles centered at C and D are given in the plane such that the inversion of circle $\bigcirc D$ about circle $\bigcirc C$ is circle $\bigcirc D$ itself. Show that the circles are either orthogonal or are the same circle.

11.) (30 pts. total) A circle centered at C has radius r, and another circle centered at K has radius a. Denote the distance between C and K as w. Assume a is less than w so that point C is exterior to circle $\bigcirc K$.
a.) Show that the inversion of circle $\bigcirc K$ about circle $\bigcirc C$ is also a circle. (25 pts.)
b.) Determine the radius of this circle. (5 pts.)

Curves
Consider a curve to be a one-dimensional, differentiable and continuous path that does not intersect itself. Every point on the curve is contained in a line tangent to the curve at that point. Assume that the inversion of a curve about a circle is also a curve.

12.) (20 pts.) Suppose two curves C_1 and C_2 in the plane intersect at exactly one point P and that both curves lie entirely in the exterior of a given circle centered at O. Show that the angle between the lines tangent to C_1 and C_2 at P is equal to the angle between the tangents to the inversions of C_1 and C_2 at P', where all inversions are taken with respect to circle $\bigcirc O$.



13.) (30 pts. total) In the figure shown above, $AB = AC = x$, and $BQ = QC = PC = BP = y$, with $x > y$. Suppose point A is fixed.
a.) Show that there exists a fixed point Z in the plane of the figure such that if $PZ = AZ$, then the locus of Q is a line segment. (25 pts.)
b.) What is the length of this line segment? (5 pts.)