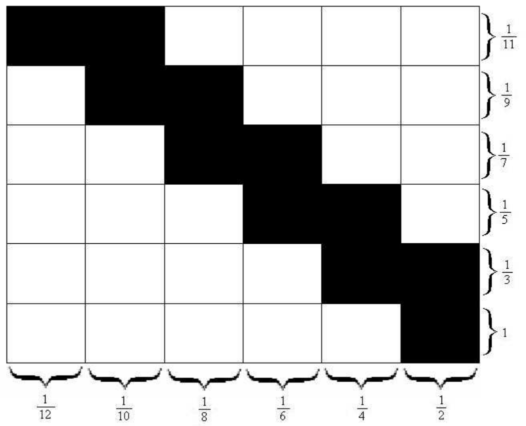
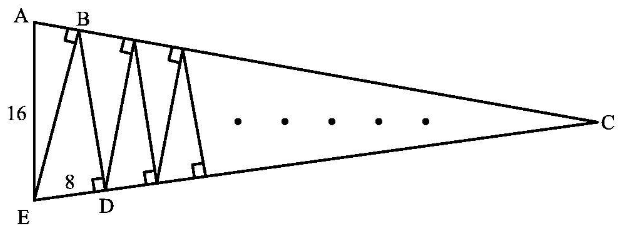
Geometry Test
2005 Rice Math Tournament
February 26, 2005

1. A dog is tied via a 30 ft. leash to one corner of a 10 ft. by 20 ft. dog pen. Given that the dog is initially on the outside of the pen and that neither he (nor his leash) can cross the pen's fence, what area of land does he have to roam in?

2. Find the area of the shaded region:

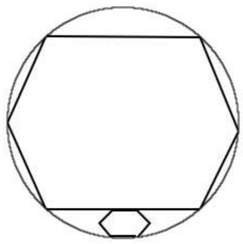


3. An infinite series of similar right triangles converges to point C. If $\overline{AE} = 16$, and $\overline{ED} = 8$, what is the sum of all the vertical segiments $(\overline{AE} + \overline{BD} + ...)$?



4. Find the center of the circle of radius 1 centered on the y-axis that is tangent to the parabola $y = x^2$ in two places.

5. A regular hexagon is inscribed in a circle with radius r. Another regular hexagon is formed with 2 vertices on an edge of the first hexagon and 2 vertices on the circle as shown below. Find the ratio of the area of the smaller hexagon to the area of the larger hexagon.

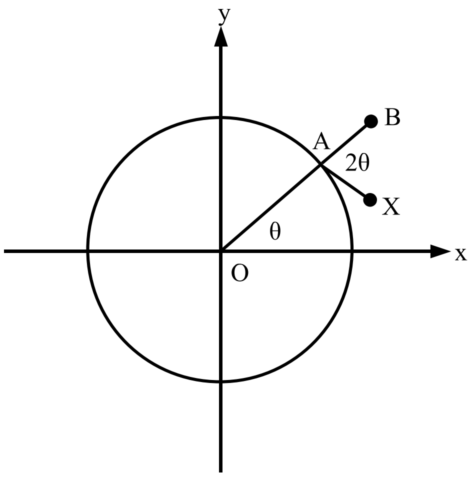


6. In circle O, chords AB and AC are drawn so that AB = AC. Chord MN is drawn intersecting AB and AC at points R and S, respectively. Given that AR = SC = 17 and RB = AS = 13, what is the maximum value of $|MR - NS|$?

7. Suppose $\triangle ABC$ is an equilateral triangle with area 1. Points Q and P are on AB and points R and S are on AC with QR, PS and BC all parallel to each other. Also, $\overline{QR} < \overline{PS}$. Points P' and S' are chosen on BC so that PP' and SS' are each perpendicular to BC. Likewise, Q' and R' are chosen on PS so that QQ' and RR' are perpendicular to PS. What is the maximum possible area enclosed by the union of the two rectangles $PP'S'S$ and $QQ'R'R$?

8. A unit lattice square in the plane is a square of side 1 whose vertices have integer coordinates. Given that $(20.05)^2 \cdot \pi \approx 1260$. Let N be the number of lattice squares that are entirely contained in a circle of radius 20.05 centered at the origin. Find $\lfloor \frac{N}{10} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. (Hint: A very good approximation for area in the plane of a smooth figure is given by $I + \frac{E}{2}$ where I is the number of unit lattice squares contained in the figure and E is the number intersected by the boundary)

9. For each θ created by line OA and the x-axis, point X is the point (-2θ) from $\overrightarrow{AB}$, where $\overline{OA} = 6$, and $\overline{AX} = 3$. Find the area enclosed by X as A takes each point along the unit circle.



10. An internal diagonal of a rectangular prism connects 2 vertices and does not lie entirely in one face. Let P be a rectangular prism with volume 1, and let l be one of its internal diagonals. Suppose C is a vertex of the prism that is not one of the vertices of l. A point K is chosen on l, and a new prism p' is formed such that $\overline{CK}$ is an internal diagonal of p', and the faces of p' are parallel to those of p. What is the maximum volume of p'?