General Solutions
2005 Rice Math Tournament
February 26, 2005

1. **Answer: 7**

   $3x + 5y = 83$, $x + y = 23$, $2y = 14$, $y = 7$, So there are seven five legged goats.

2. **Answer: $\frac{\pi}{2}$**

   Its pretty obvious that a circle that intersects the lattice point vertices of a unit square will have the optimal area. The answer is $\frac{\pi}{2}$.

3. **Answer: 2**

   By definition

4. **Answer: 50**

   Number with both = total combinations - without 2 - without 1 + 1. This is $3^4 - 2^4 - 2^4 + 1 = 50$.

5. **Answer: 71**

   Can be found by trial and error or using the formula $m \cdot n - m - n$ for m, n relatively prime.

6. **Answer: *Sunday***

   Febuary 26, 2005 is a Saturday. It is 285 days until December 8, 2005. Since 280 is a multiple of 7, that will be a Thursday. 2019 is 14 years from then and that will be a multiple of 7 (14×365) plus the number of leap years which is 3. Thus December 8, 2019 will be a Sunday.

7. **Answer: $\frac{7}{160}$**

   If the 4-sided die shows a 1 there are 19 different rolls of the first 20-sided die that can still achieve a sum of 21. If the 4-sided die shows a 2 there are 18. If it shows a 3 there are 17, and if a 4, there are 16. Thus, there are $19 + 18 + 17 + 16 = 70$ ways to get 21.

   $$\frac{70}{20 \cdot 20 \cdot 4} = \frac{7}{160}$$

8. **Answer: 165**

   The number of 0's is the number of factors of 26 there are. As $26 = 13 \times 2$, and there are more factors of 13 than 2's, we need only find the number of factor's of 13 in 2005!. $2005 \div 13 > 154$; $2005 \div 169 > 11$; $154 + 11 = 165$

9. **Answer: m^2**

   A lillypad will be off if an even number of frogs jump on it. Hence, the m^{th} lillypad will be on if m has an odd number of factors. m has an odd number of factors $\Longleftrightarrow m$ is a perfect square.

10. **Answer: 15**

    All such parallelograms have 2 vertices with $y > 0$. There are 6 points that satisfy this so there are $\binom{6}{2=15}$ ways to pick them. Each choice uniquely determines the other 2 points of the parallelogram, so the answer is 15.

11. **Answer: $\frac{18!}{2!2!3!} = \frac{18!}{24}$**

    The numerator denotes the number of arrangements of 18 different words. The denominator divides out redundancies since "in," "the," and "words" are repeated. Note from author: I am purposely not capitalizing the first letter of the sentence as this might otherwise lead to ambiguity.

12. **Answer: 800π**

$$\frac{3}{4} \cdot 30^2\pi + \frac{1}{4} \cdot 10^2\pi + \frac{1}{4} \cdot 20^2\pi = 800\pi$$

13. **Answer: 61**

Of the 4 consecutive integers two must be odd and one must be of the form 4m+2. Therefore for 128 to divide the product, the remaining integer must be a multiple of 64. Hence the least such n is n=61.

14. **Answer: 32**

Since all triangles are similar, $\overline{AE} : \overline{EB} = \overline{EB} : \overline{ED}$. Let $\frac{EB}{AE} = x$. Since $\overline{EB}^2 = \overline{ED}^2 + \overline{BD}^2$, $(16x)^2 = 8^2 + (16x^2)^2$, $\therefore x = \frac{\sqrt{2}}{2}$. Since $\overline{AE} = 16$, $\overline{BD} = 16x^2$, the next vertical segment is $16x^2 \times x^2$, and so on. $\therefore$ sum of all vertical segments is a geometric series $\frac{16}{1-x^2} = 32$.

15. **Answer: $\frac{19}{96}$**

Consider the result of each toss mod 5. We have a success if we roll:
$(1,4)$ with probability $\frac{3}{12} \cdot \frac{1}{8} = \frac{3}{96}$
$(2,3)$ with probability $\frac{3}{12} \cdot \frac{2}{8} = \frac{6}{96}$
$(3,2)$ with probability $\frac{2}{12} \cdot \frac{2}{8} = \frac{4}{96}$
$(4,1)$ with probability $\frac{2}{12} \cdot \frac{2}{8} = \frac{4}{96}$
$(0,0)$ with probability $\frac{2}{12} \cdot \frac{1}{8} = \frac{2}{96}$
The total chances are $\frac{19}{96}$.

16. **Answer:** The coefficient of x^{1000} in $(1 + x^2 - x^5)^{2005}$

x^{1000} is even, so its coefficient is unchanged under the transformation $x \to -x$.
Under this transformation,
$(1 - x^2 + x^5)^{2005} \to (1 - x^2 - x^5)^{2005}$ and
$(1 + x^2 - x^5)^{2005} \to (1 + x^2 + x^5)^{2005}$.
After expanding $(1 + x^2 + x^5)^{2005}$, the x^{1000} will all have positive coefficients.
However, there will be x^{1000} terms in the expansion of $(1 - x^2 - x^5)^{2005}$ with negative coefficients.

17. **Answer: 840**

There are 6! ways to order the numbers 1-6. Half will have the 1 before the 2 and half will have the 2 before the 1. Thus, the number of ways to stack the leftmost six blocks is $\frac{6!}{2^3}$. Now we must simply determine in which three of the nine positions the sevens could have been read. There are $\binom{9}{3}$ ways to do this. Thus, our final answer is $\frac{6!}{2^3} \cdot 9C3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2^3} \cdot \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$.

18. **Answer: 12**

There are 10 powers of 4 (1, 4, ..., 4^9) and 8 powers of 6 (1, 6, ..., 6^7), but we have counted powers of 12 twice and there are 6 of these. The inclusion-exclusion principle tells us that the answer is $10 + 8 - 6 = 12$ satisfactory numbers.

19. **Answer: $625ft$**

The walls move $40\frac{ft}{sec}$ total taking 25 sec. to close. Therefore, the ball travels $25 \times 25 = 625ft$.

20. **Answer: 36**

Let n be an integer with $1 \leq n \leq 100$, and say $n = a^b$ where a is not a perfect power. Then $n^n = (a^b)^n = a^{bn}$. Since a is not a perfect power, if n^n is a perfect cube then bn is divisible by 3, so either b or n is divisible by 3. If b is divisible by3, $n = a^b$ is a perfect cube, s n = 1, 8, 27, and 64. If n is divisible by 3, then n = 3, 6, 9, ..., 99. This is a total of 4+33=37 n-values, but $n = 27$ was counted twice, so there are 36 perfect cubes.

21. **Answer: $\mathbf{247 = 30 + 42 + 70 + 105}$**

No set of less than four containers works, since any three of them have at least one prime factor in common. To measure 1, we can measure 6 with the 30 and 42 containers, then 2 with the 6 we found plus the 70, and finally 1 when we add the 105 to the mix.

22. **Answer: $\mathbf{\frac{3}{4}}$**

From case by case analysis, the sides are found to be: $x + y = \frac{1}{2}$, $x + y = -\frac{1}{2}$, $x = \pm\frac{1}{2}$, and $y = \pm\frac{1}{2}$. This forms a hexagon which can be split up into two squares and two isosceles right triangles, with side/leg length $\frac{1}{2}$. Thus the area is $2(\frac{1}{2})^2 + 2 \cdot \frac{1}{2}(\frac{1}{2})^2 = 3(\frac{1}{2})^2 = \frac{3}{4}$.

23. **Answer: 3**

The roots are n, $n+1$, and $n+2$. Then $a^2 = ((n) + (n+1) + (n+2))^2 = (3n+3)^2 = 9(n+1)^2$. $b + 1 = n(n+1) + n(n+2) + (n+1)(n+2) + 1 = 3n^2 + 6n + 3 = 3(n+1)^2$. So $\frac{a^2}{b+1} = 3$.

24. **Answer: $\mathbf{5 \cdot \frac{7!}{4} \cdot \frac{4!}{2} = 15 \cdot 7!}$**

The orderings of the vowels and consonants are independent of how the vowels and consonants are ordered relative to each other. So first we compute the number of ways to arrange the vowels and consonants. Since the 4 V's partition the 7 C's into 5 possibly empty blocks. Since the maximum these blocks can have to satisfy the rules are 1,2,2,2, and 1, so exactly one of them must be one short. There are 5 ways to do this. Then we multiply the number of permutations of the vowels AEAI and consonants MTHMTCS, of which there are $\frac{7!}{4} \cdot \frac{4!}{2}$.

25. **Answer: 8**

$$a_1 = 0$$
$$a_2 = \frac{1}{4}$$
$$a_n = a_{n-1} + \frac{1}{4}$$
$$a_n = \frac{1}{4}(n-1)$$
$$\frac{1}{4}(33-1) = \frac{1}{4}(32) = 8$$