Calculus Solutions
2005 Rice Math Tournament
February 26, 2005

1. **Answer: 17**

   The profit is P is revenue minus cost. Thus $P = n * D - (600 + 10n + n^2) = -3n^2 + 100n - 600$. Consider the smoothed version of P, and we can find the critical point of $16\frac{2}{3}$. Thus either $n = 16$ or $n = 17$ is the largest profit. Quick computation yields $n = 17$.

2. **Answer: 0**

   $f(0) = \frac{1}{2}f(1)$, $f(-\frac{1}{2}) = \frac{1}{2}f(0)$, $f(-\frac{3}{4}) = \frac{1}{2}f(-\frac{1}{2})$, and so on. So the limit is 0.

3. **Answer: $(\frac{24}{4+\pi}, \frac{24}{4+\pi})$**

   We want to maximize the area of the window. Let x be the radius of the semicircular region and y the height of the rectangular portion of the window. The area of the window is $A = \frac{1}{2}\pi x^2 + 2xy$. The perimeter is $P = \pi x + 2x + 2y$. Eliminating y in A yields $A = 24x - (2 + \frac{\pi}{2})x^2$. We can take the derivative and find the critical point of $x = \frac{24}{4+\pi}$. Solving for y yields the same answer.

4. **Answer: $\frac{1}{13}$**

   Applying the chain rule to $f(f^{-1}(x)) = x$ yields $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$.
   Rearranging yields $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.
   $f^{-1}(-5)$ is the solution to $-5 = x^5 + 2x^3 + 2x$ which has the unique solution of $x = -1$.
   $f'(-1) = 5 + 6 + 2 = 13$.
   Thus $\frac{1}{13}$ is the answer.

5. **Answer: $\frac{257}{20}$**

   The average value is $\frac{\int_{-1}^{4} f(x)dx}{4-(-1)} = \frac{1}{5}\int_{-1}^{4} |x^3|dx = \frac{1}{5}(\int_{-1}^{0} -x^3 dx + \int_{0}^{4} x^3 dx) = \frac{257}{20}$.

6. **Answer: $6\sqrt{6}$ feet**

   From the diagram below, we can see we want to maximize θ. Note that $\tan\theta = \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{36}{x} - \frac{6}{x}}{1 + \frac{36}{x}\frac{6}{x}} = \frac{42x}{x^2+216}$. Using implicit differentiation, we can find that $\theta' = \cos^2\theta \cdot \frac{-42x^2+9072}{(x^2+216)^2}$. The maximum occurs at $x = 6\sqrt{6}$.

7. **Answer: $\frac{1}{e}$**

   $$f(x) = x^x$$
   $$\Longrightarrow \ln f(x) = x \ln x$$
   $$\Longrightarrow \frac{f'(x)}{f(x)} = 1 + \ln x$$
   $$\Longrightarrow f'(x) = f(x)[1 + \ln x] = x^x[1 + \ln x]$$

   Critical points at $f'(x) = 0$

   $$\Longrightarrow 0 = x^x[1 + \ln x]$$
   $$\Longrightarrow 0 = 1 + \ln x \ \ (x^x \textit{ always positive for } x > 0)$$
   $$\Longrightarrow \ln x = -1$$
   $$\Longrightarrow x = e^{-1} = \frac{1}{e} \ \ (\textit{critical point})$$

Furthermore,

$$f''(x) = f'(x)[1 + \ln x] + \frac{f(x)}{x}$$

$$f'' = x^x(1 + \ln x)^2 + x^{x-1} \ > \ 0 \ \ (for\ x > 0)$$

So, $f''(x) > 0$ for all $x > 0$.
Hence, the function is concave up everywhere, and so $x = \frac{1}{e}$ must be a minimum.

8. **Answer:** $\boldsymbol{e^2 - 1}$

$$\begin{aligned} \lim_{n\to\infty} \int_0^2 \left(1 + \tfrac{t}{n+1}\right)^n dt \quad &= \lim_{n\to\infty} \left(1 + \tfrac{t}{n+1}\right)^{n+1} |_0^2 \\ &= \lim_{n\to\infty} \left(1 + \tfrac{2}{n+1}\right)^{n+1} - \lim_{n\to\infty} \left(1 + \tfrac{0}{n+1}\right)^{n+1} \\ &= e^2 - 1 \text{ (since } \lim_{n\to\infty} \left(1 + \tfrac{t}{n}\right)^n = e^t) \end{aligned}$$

9. **Answer:** $\boldsymbol{-1}$

Use integration by parts:

$$\int_{-\infty}^{\infty} \delta'(x) \sin(x) dx$$

$$= \delta(x) \sin(x) dx|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) \cos(x) dx$$

$$= 0 - cos(0) \quad (\delta(x) = 0 \text{``}at'' \pm \infty)$$

$$= -1$$

10. **Answer:** $\boldsymbol{\frac{2}{p(1-p)^3}}$

$$\sum_{i=0}^{\infty} i(i+1)p^{i-2} = \sum_{i=0}^{\infty} i(i-1)p^{i-2} + 2\sum_{i=0}^{\infty} ip^{i-2}$$

$$\sum_{i=0}^{\infty} i(i-1)p^{i-2} = \frac{d^2}{dp^2} \sum_{i=0}^{\infty} p^i = \frac{d^2}{dp^2} \frac{1}{1-p} = \frac{d}{dp} \frac{1}{(1-p)^2} = \frac{2}{(1-p)^3}$$

$$\sum_{i=0}^{\infty} ip^{i-2} = \frac{1}{p} \sum_{i=0}^{\infty} ip^{i-1} = \frac{1}{p} \frac{d}{dp} \frac{1}{1-p} = \frac{1}{p(1-p)^2}$$

Then $\frac{2}{(1-p)^3} + \frac{2}{p(1-p)^2} = \frac{2}{p(1-p)^3}$