Calculus Test
2005 Rice Math Tournament
February 26, 2005

1. Mathisgreatco, Inc. can produce at most 24 spherical cow statues each week. Experience has shown that the demand for spherical cows sets the price at $D = 110 - 2n$ where n is the number of statues produced that week. Producing n statues costs $600 + 10n + n^2$ dollars. How many statues should be made each week to maximize profit?

2. $f(x) = \frac{1}{2}f(2x+1)$ and $f(1) = 2005$. What is $\lim_{x\to -1^+} f(x)$ assuming it exists?

3. Sammy the Owl wants to design a window that is a rectangle with a semicircle on top. If the total perimeter is constrained to be 24 feet, what dimensions should Sammy pick so that the window admits the greatest amount of light? Give the radius of the semicircular region and the height of the rectangular portion.

4. Given $f(x) = x^5 + 2x^3 + 2x$, find $(f^{-1})'(-5)$.

5. Find the average value of $f(x) = |x^3|$ on the interval $[-1, 4]$.

6. Sammy the Owl wants to watch his home movie of a trip to Rice University on his movie screen. The lower edge of the screen, which is 30 feet high, is 6 feet above eye level. How far from the screen should the observer sit to obtain the most favorable view, i.e. to maximize the visual angle (the observed angle between the top and bottom of the screen)?

7. Find the value of x for which the function $f(x) = x^x$ achieves a minimum on the positive real line (i.e. for all real numbers x such that $x > 0$)?

8. Evaluate
$$\lim_{n\to\infty} \int_0^2 \left(1 + \frac{t}{n+1}\right)^n dt$$

9. Define a new "function" $\delta(x)$ with the properties $\delta(x) = 0$ for all $x \neq 0$ and
$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$
for every function f(x). Compute
$$\int_{-\infty}^{\infty} \delta'(x) \sin(x) dx.$$
For the purposes of this problem, you may assume the existence of
$$\delta'(x) \equiv \frac{d\,\delta(x)}{dx}.$$

10. Compute
$$\sum_{i=0}^{\infty} i(i+1)p^{i-2}$$
for $0 < p < 1$.