Algebra Solutions
2005 Rice Math Tournament
February 26, 2005

1. **Answer:** $\mathbf{\frac{1}{3}}$

   If g is the fraction of the class that are girls and b is the fraction of the class that are boys, then $91g + 85b = 89$ and $g + b = 1$, so $91(1 - b) + 85b = 89$, which simplifies to $b = \frac{1}{3}$.

2. **Answer: 1**

$$(x^2+4)^2 = (2x-3)^2$$
$$(x^2+4)^2 - (2x-3)^2 = 0$$
$$(x^2+4+2x-3)(x^2+4-2x+3) = 0$$
$$(x^2+2x+1)(x^2-2x+7) = 0$$
$$(x+1)^2(x^2-2x+7) = 0$$

   $x^2 - 2x + 7$ has no real roots since $2^2 - 4 \cdot 7 < 0$.

3. **Answer:** $\boldsymbol{m^2}$

   A lillypad will be off if an even number of frogs jump on it. Hence, the m^{th} lillypad will be on if m has an odd number of factors. m has an odd number of factors $\Longleftrightarrow m$ is a perfect square.

4. **Answer:** $\mathbf{\{(2,4),(4,2)\}}$

$$\frac{\log A}{\log B} = \frac{A}{B}$$
$$B \log A = A \log B$$
$$\log A^B = \log B^A$$
$$A^B = B^A$$
$$(A, B) \in \{(2,4),(4,2)\}$$

5. **Answer: 17**

$$\begin{aligned} \text{f}(0) &= \text{f}(-8+8) \\ &= \text{f}(-8) + \text{g}(8) + 8 \\ &= -8 + 17 + 8 \\ &= 17 \end{aligned}$$

6. **Answer: 2005**

   $x = 2$ so $(x^2+1)\Big(\big((x^2+1)\cdot x^2\big)^2 + 1\Big) = 2005$.

7. **Answer: 3**

   The roots are n, $n+1$, and $n+2$. Then $a^2 = ((n) + (n+1) + (n+2))^2 = (3n+3)^2 = 9(n+1)^2$. $b + 1 = n(n+1) + n(n+2) + (n+1)(n+2) + 1 = 3n^2 + 6n + 3 = 3(n+1)^2$. So $\frac{a^2}{b+1} = 3$.

8. **Answer: $a = 4, b = 3$**

Since $24ab32$ is divisible by 9,

$$2 + 4 + a + b + 3 + 2 \equiv 0 (mod 9)$$

$$a + b \equiv -2 (mod 9)$$

So we have either $a + b = 7$ or $a + b = 16$
Since $24ab32$ is divisible by 11,

$$-2 + 4 - a + b - 3 + 2 \equiv 0 (mod 11)$$

$$a - b \equiv 1 (mod 11)$$

So $a - b = 1$. Hence, the only solution is $a = 4, b = 3$.

9. **Answer: 12**

Let a, b, c be the roots of $x^3 + Ax^2 + Bx + C = 0$.

$$A = -(a + b + c) = 1.$$

$$B = ab + bc + ac = \frac{1}{2}\left[(a + b + c)^2 - (a^2 + b^2 + c^2)\right] = \frac{1}{2}[1 - 17] = -8$$

Therefore:

$$a^3 + Aa^2 + Ba + C = 0$$

$$b^3 + Ab^2 + Bb + C = 0$$

$$c^3 + Ac^2 + Bc + C = 0$$

Add them up to get:

$$11 + A(17) + B(-1) + C = 0$$

$$11 + 17 + 8 + 3C = 0$$

$$C = \frac{-36}{3} = -12$$

$$C = -abc.$$

Thus,

$$abc = 12.$$

Alternatively, solve to get

$$a = b = -2,\ c = 3.$$

10. **Answer: $x^4 - 14x^2 + 9$**

Let $x = \sqrt{2} + \sqrt{5}$.
Then $x^2 = 2\sqrt{10} + 7$
$x^4 = 89 + 28\sqrt{10}$
$x^4 - 14x^2 = -9$
so if $p(x) = x^4 - 14x^2 + 9$, then $p(\sqrt{2} + \sqrt{5}) = 0$.
$p(x)$ must be of at least degree 4 since $\sqrt{2} + \sqrt{5}$ is not the root of any quadratic or cubic polynomials with integer coefficients.