Advanced Solution
2005 Rice Math Tournament
February 26, 2005

1. **Answer: $\frac{9}{49}$**

$$\begin{aligned} P\left(True|Droop\right) &= \frac{P(True)P(Droop|True)}{P(Droop)} \\ &= \frac{P(T)P(D|T)}{P(T)(P(D|T))+P(lie)P(D|lie)} \\ &= \frac{\frac{3}{5}\frac{1}{10}}{\frac{3}{5}\frac{1}{10}+\frac{2}{5}\frac{2}{3}} = \frac{9}{49} \end{aligned}$$

2. **Answer: 3125**

$$(2000+5)^{2005} = 2000^k \cdot 5^m + 5^{2005}$$

Since $2000^k \cdot 5^m$ is divisible by 10,000, we want to find the last digits of 5^{2005}.
[The following are all mod 10,000.]
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$
$5^5 = 3125$
$5^6 = 5625$
$5^7 = 8125$
$5^8 = 625 = 5^4$
...so these repeat every four starting with 5^3.

$$(2005-3)\ (mod\ 4) = 2002\ (mod\ 4) = 2$$

Therefore, $5^{2005} = 3125$.

3. **Answer:**

4. **Answer: $\frac{7}{2} - \frac{\sqrt{3}}{2}$**

The square projects out of the hexagon on top and bottom in isosceles right triangles. The area of the hexagon is $\frac{3s^2\sqrt{3}}{2}$ where $s = 1$. Each triangle is easiest dealt with as two triangles: $A = \frac{3\sqrt{3}}{2} + 2 \cdot 2 \cdot \frac{1}{2}(1 - \frac{\sqrt{3}}{2})^2 = \frac{3\sqrt{3}}{2} + 2 - 2\sqrt{3} + \frac{3}{2} = \frac{7}{2} - \frac{\sqrt{3}}{2}$

5. **Answer: 500**

$$\lfloor \frac{2005}{5} \rfloor = 401$$

$$\lfloor \frac{401}{5} \rfloor = 80$$

$$\lfloor \frac{80}{5} \rfloor = 16$$

$$\lfloor \frac{16}{5} \rfloor = 3$$

$$\lfloor \frac{3}{5} \rfloor = 0$$

$$401 + 80 + 16 + 3 = 500$$

6. **Answer:** $\frac{\sqrt{2005\cdot 2009}-2005}{2}$

Let the continued fraction be x.

$$x = \frac{2005}{2005+x}$$
$$x^2 + 2005x - 2005 = 0$$
$$x = \frac{-2005 \pm \sqrt{2005^2 + 4\cdot 2005}}{2}$$
$$x = \frac{\sqrt{2005\cdot 2009} - 2005}{2}$$

Note: (-) is dropped since clearly positive.

7. **Answer:** $\frac{31}{30}$ muffins

Let $\phi(n)$ = number of integers relatively prime to n.

$$\phi(n) = n\Big(1-\frac{1}{p_1}\Big)\Big(1-\frac{1}{p_2}\Big)\dots\Big(1-\frac{1}{p_k}\Big),$$

where $p_1, \dots, p_k$ are distinct primes dividing n.

We need $\phi(n) = 4$. So $5, 8, 10, 12$ are the only solutions to this equation.

Thus your expected winnings are

$$\frac{1}{4}\cdot\frac{4}{5}\cdot 3 - \frac{1}{4}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{4}{8}\cdot 3 - \frac{1}{4}\cdot\frac{4}{8} + \frac{1}{4}\cdot\frac{4}{10}\cdot 3 - \frac{1}{4}\cdot\frac{6}{10} + \frac{1}{4}\cdot\frac{4}{12}\cdot 3 - \frac{1}{4}\cdot\frac{8}{12}$$
$$= \frac{3}{5} - \frac{1}{20} + \frac{3}{8} - \frac{1}{8} + \frac{3}{10} - \frac{3}{20} + \frac{1}{4} - \frac{1}{6} = \frac{72-6+45-15+36-18+30-20}{120} = \frac{124}{120} = \frac{31}{30}$$

8. **Answer:** $\frac{19}{20}$

If $P(x) = \sum_{i=0}^{n} c_i x^i$, then $P(x) - P(y) = \sum_{i=0}^{n} c_i\left(x^i - y^i\right)$. Note that $x^i - y^i$ is divisible by $x-y$. If $x - y \geq 2$, then $P(x) - P(y)$ will be composite. Since the degree is at least 2, $P(x) - P(y) > c_2(x^2 - y^2) = c_2(x+y)(x-y)$. Note that $x+y > 1$, so $\frac{P(x)-P(y)}{x-y}$ is an integer larger than 1. So we only need $x - y \geq 2$. There are 780 total pairs (x, y). All will work except $(x, y) = (2, 1), (3, 2), \dots, (40, 39)$. The answer is $\frac{780-39}{780} = \frac{19\cdot 39}{20\cdot 39} = \frac{19}{20}$.

9. **Answer:** $S_{2005,1}, S_{2002,2}, S_{399,5}$, and $S_{196,10}$

$$\sum_{n\in S_{mk}} n = \sum_{i=0}^{k-1} m + i$$
$$2005 = \frac{(k-1)(k)}{2} + km$$
$$4010 = (k-1)(k) + 2km$$
$$4010 = k(k-1+2m)$$

k and $k-1+2m$ must be factors of 4010

$4010 = 2\cdot 5\cdot 401$

$k = 1$ yields $m = 2005$

$k = 2$ yields $m = 1002$

$k = 5$ yields $m = 399$

$k = 10$ yields $m = 196$

For $k = 401$, we get $10 = 2m + 400$, which has no positive integers solutions for m.

Thus, $k = 1, 2, 5, 10$ are the only solutions.

10. **Answer: $\mathbf{(2,8),(2,12),(4,8),(4,12),(6,8),(6,12),(8,8),(8,12)}$**

$$5^m + 3^n - 1 \equiv 0 \; (mod\; 15)$$

Taking mod 5:

$$3^n - 1 \equiv 0 \; (mod\; 5)$$

$$n \equiv 0 \; (mod\; 4)$$

Taking mod 3:

$$5^m - 1 \equiv 0 \; (mod\; 3)$$

$$m \equiv 0 \; (mod\; 2)$$

Indeed,

$$5^{2p} + 3^{4q} - 1 \equiv 10^p + 6^q - 1 \equiv 10 + 6 - 1 \equiv 0 \; (mod\; 15).$$

Hence, all solutions are of the form

$$m = 2p, n = 4q, p > 0, q > 0.$$