

POWER SOLUTIONS  
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Much of the material on this test was taken from the book Combinatorics: A Problem Oriented Approach  
by Daniel A. Marcus (Mathematical Association of America, 1998)

*Combinatorial Arguments*

1. **Answer:**  $3^{100}$

We can look at this as the number of ways of assigning one of 3 colors to each of 100 elements. First, we simply pick which of the 3 colors map to each element, and there are  $3^{100}$  ways of doing this. Another way to do this is to first pick an integer  $n$  between 0 and 100 to be the number of elements that will be colored with colors #2 & #3. There are  $\binom{100}{n}$  ways to choose these  $n$  elements and  $2^n$  ways to color them (since we must assign one of #2 and #3 to each). The remaining  $100 - n$  elements must be assigned color #1. Thus,  $\sum_{n=0}^{100} \binom{100}{n} \cdot 2^n$  is also the number of ways to assign one of 3 colors to each of 100 elements. So, we conclude that these two are equal.

2. **Answer:**  $\binom{35}{5} = \binom{35}{30} = 324632$

Assume each combination is ordered in numerical order. Since repetition is allowed, we simply have to choose the number of times each element is repeated (note that 0 repetitions is allowed). One way of quickly counting this is known as the “stars and bars” argument. We consider 30 identical “stars” in a row, and we place 5 “bars” between the stars. Multiple bars between two stars are allowed.

The number of locations for the bars is equal to the number of 30-digit combinations from our set - simply let the number of 0s taken be the number of stars before the first bar, the number of 1s taken be the number of stars between the first and second bars, ..., the number of 5s be the number of stars after the fifth bar. Also, there are clearly  $\binom{35}{5}$  ways to place the bars between the stars (consider 35 slots and put the bars in 5 of them). So, there are  $\binom{35}{5}$  30-digit combinations with repetition from  $\{0, 1, 2, 3, 4, 5\}$ .

3. **Answer:**  $\binom{20}{5} = \binom{20}{15}$

We use a similar approach as in the preceding argument, but there are more constraints this time. So, in our combination, first take 0 0s, 1 1s, 2 2s, ..., and 5 5s. This accounts for 15 digits and leaves 5 digits left to choose from  $\{0, 1, 2, 3, 4, 5\}$ , and by the argument above there are  $\binom{20}{5}$  ways to do this.

4. **Answer:**  $\binom{n+1}{k+1}$

We can look at this as the number of ways to choose a  $k + 1$  element subset from  $\{1, 2, \dots, n + 1\}$ . Clearly, there are  $\binom{n+1}{k+1}$  ways to do this. Another way to count the number of  $k + 1$  element subsets is to consider the largest element in the subset. Say this largest element is  $i$ ; then you need to choose  $k + 1 - 1 = k$  elements in  $\{1, 2, \dots, i - 1\}$ , and there are  $\binom{i-1}{k}$  ways to do this. Since  $i$  can range from  $k + 1$  to  $n + 1$ , we get that  $\binom{n+1}{k+1} = \sum_{i=k+1}^{n+1} \binom{i-1}{k} = \binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k}$ , as desired.

5. We show these are both ways of counting the number of ways to pick from  $n$  people a committee of  $m$  people and, from that committee, a subcommittee of  $k$  people.

The first way to select this committee and subcommittee is to first select the committee from the group of  $n$  people (there are  $\binom{n}{m}$  ways to do this), and then select the subcommittee from the committee (there are  $\binom{m}{k}$  ways to do this), thus giving a total number of ways as  $\binom{n}{m} \binom{m}{k}$ .

Another way to do this is to first select the members of the group of  $n$  people that will be on the subcommittee (there are  $\binom{n}{k}$  ways to do this), and then select the remaining  $m - k$  members of the committee from the  $n - k$  remaining members of the group (there are  $\binom{n-k}{m-k}$  ways to do this). This gives the total number of ways as  $\binom{n}{k} \binom{n-k}{m-k}$ .

Since these are two different ways of counting the same thing, we conclude that they are equal.

6. **Answer:**  $\binom{50}{20} \cdot 2^{20} = \binom{50}{30} \cdot 2^{20}$

By the equation in the last problem, this sum is equal to  $\sum_{i=0}^{20} \binom{50}{20} \binom{20}{i}$ , which is equal to  $\binom{50}{20} \sum_{i=0}^{20} \binom{20}{i}$ . We now show that  $\sum_{i=0}^{20} \binom{20}{i} = 2^{20}$  by counting the number of subsets of  $\{1, 2, \dots, 20\}$ . Clearly, we can simply choose for each element whether it is in the subset or not, which gives a total of  $2^{20}$  subsets. Also, we can first decide to count subsets of size  $i$ , and then pick which elements are in the subset - there are  $\binom{20}{i}$  ways to do this. Since  $i$  can range from 0 to 20, we conclude that there are  $\sum_{i=0}^{20} \binom{20}{i}$  subsets of  $\{1, 2, \dots, 20\}$ , and so  $\sum_{i=0}^{20} \binom{20}{i} = 2^{20}$  as desired. So, we conclude that  $\sum_{i=0}^{20} \binom{50}{i} \binom{50-i}{20-i}$  is equal to  $\binom{50}{20} 2^{20}$ .

7. **Answer:**  $100 \cdot 2^{99} = 2^{101} \cdot 5^2$

We can look at this as the number of ways of picking a nonempty subset of  $\{1, 2, \dots, 100\}$  where one element in this subset is “special”. One way to do this is to simply pick the “special” element first (100 ways to do this), and pick a subset of the remaining 99 elements to be the rest of the subset ( $2^{99}$  ways to do this), for a total of  $100 \cdot 2^{99}$  ways. Another way to do this is to first decide on the total size of the subset - say it is of size  $i$  (clearly  $1 \leq i \leq 100$ ). Then, pick the subset ( $\binom{100}{i}$  ways to do this), and from the subset, pick the “special” element ( $i$  ways to do this), for a total of  $i \cdot \binom{100}{i}$ . So, we sum over all possible values of  $i$  and get  $\sum_{i=1}^{100} i \cdot \binom{100}{i}$  to be the total number of ways. So, this is equal to  $100 \cdot 2^{99}$ .

### *Stirling Numbers*

8. (a) **Answer:** 6

(b) **Answer:** 15

(c) **Answer:** 31

9. **Answer:**  $S(m, 2) = 2^{m-1} - 1$

$S(m, 2)$  is the number of ways to break an  $m$ -element set into 2 nonempty parts. Let’s say this set is  $\{1, 2, \dots, m\}$ . So, first we pick a subset of  $\{1, 2, \dots, m\}$  to be the first part - there are  $2^m$  ways to do this. However, we can’t pick the empty set, and we can’t pick the whole set (since then the other part would be empty), so we have  $2^m - 2$  choices. Then, the second part must be the complement of the first part. However, order between the parts doesn’t matter, so we’ve counted every possible partition twice, and so the final answer is  $\frac{2^m - 2}{2} = 2^{m-1} - 1$ .

10. **Answer:**  $S(m, m - 1) = \binom{m}{2} = \frac{m(m-1)}{2}$

$S(m, m - 1)$  is the number of ways to break an  $m$ -element set into  $m - 1$  nonempty parts. Since each part is nonempty, clearly the only way to break the set into  $m - 1$  parts is to have one part that has exactly 2 elements, and the rest of the parts have exactly one element. So, the number of ways to choose the part with 2 elements is  $\binom{m}{2}$ , and the rest of the parts are determined from this (since the order of the parts does not matter). Thus,  $S(m, m - 1) = \binom{m}{2}$ .

11. **Answer:**  $S(m, n) = S(m - 1, n - 1) + nS(m - 1, n)$

Consider an  $m$ -element set that we want to break into  $n$  nonempty parts. Let  $x$  be an arbitrary element of the set. So,  $x$  can either be in its own part, or it can be in a part with other elements. If  $x$  is on its own, the number of ways to break up the other  $m - 1$  elements into  $n - 1$  nonempty parts is  $S(m - 1, n - 1)$ , since the order of the parts does not matter. If  $x$  is not on its own, then we can first break up the other  $m - 1$  elements into  $n$  nonempty parts, and choose which one of these parts to add  $x$  to. There are  $S(m - 1, n)$  ways to break up the elements, and  $n$  parts  $x$  has to choose from, so that makes a total of  $nS(m - 1, n)$  ways for  $x$  to not be on its own. Thus, the total number of ways to break an  $m$ -element set into  $n$  nonempty parts is  $S(m - 1, n - 1) + nS(m - 1, n)$ , and so  $S(m, n) = S(m - 1, n - 1) + nS(m - 1, n)$ .

### *Card Probability*

12. **Answer:**  $\frac{\binom{24}{13}}{\binom{26}{13}} = \frac{6}{25}$

There are 24 cards that are not threes and she gets a hand of 13 cards giving an answer of  $\frac{\binom{24}{13}}{\binom{26}{13}}$ .

13. **Answer:**  $\frac{2 \cdot \binom{24}{13}}{\binom{26}{13}} = \frac{12}{25}$

The probability of either player getting no threes is still  $\frac{\binom{24}{13}}{\binom{26}{13}}$ . It is impossible for both Alice and Bob to get no threes, thus the two events are disjoint and the total probability is twice this, or  $\frac{2 \cdot \binom{24}{13}}{\binom{26}{13}}$ .

14. **Answer:**  $\frac{3 \cdot \binom{36}{13}}{\binom{39}{13}} - \frac{3 \cdot \binom{36}{10}}{\binom{39}{13}}$

Since the events of each player having no threes are not disjoint we need to use Inclusion-Exclusion. The probability of any particular player having no threes is  $\frac{\binom{36}{13} \binom{26}{13}}{\binom{39}{13} \binom{26}{13}} = \frac{\binom{36}{13}}{\binom{39}{13}}$ . The probability that two particular players have no threes is the same as one particular player having all the threes, which is  $\frac{\binom{36}{10} \binom{26}{13}}{\binom{39}{13} \binom{26}{13}} = \frac{\binom{36}{10}}{\binom{39}{13}}$ . Thus, our answer is  $\frac{3 \cdot \binom{36}{13}}{\binom{39}{13}} - \frac{3 \cdot \binom{36}{10}}{\binom{39}{13}}$ .

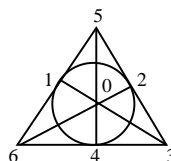
15. She wants fewer players. The probability of Alice getting no threes is a monotonically increasing function of  $N$ . For  $N$  players, this probability is  $\frac{\binom{12N}{13}}{\binom{13N}{13}}$ . To prove this function is increasing we need to show that  $\frac{\binom{12N}{13}}{\binom{13N}{13}} > \frac{\binom{12(N-1)}{13}}{\binom{13(N-1)}{13}}$ . Writing this in terms of factorials and canceling yields the following inequality, which we need to show:  $\frac{(12N)(12N-1)\dots(12N-12)}{(13N)(13N-1)\dots(13N-12)} > \frac{(12(N-1))(12(N-1)-1)\dots(12(N-1)-12)}{(13(N-1))(13(N-1)-1)\dots(13(N-1)-12)}$ . Actually, we can easily show the even stronger statement that for  $0 \leq i \leq 12(N-1)$ ,  $\frac{12N-i}{13N-i} \geq \frac{12(N-1)-i}{13(N-1)-i}$ . Equality only holds when  $i = 0$ . (To prove this stronger statement, one can clear the denominators and show the remaining fraction is a true statement.) Since the strict inequality we want to prove is a product of 13 pairs of the smaller inequalities, we are done.

16. First, in the previous question we showed that the probability is an increasing function of  $N$ . Thus, since a probability is always bounded above by 1, the probability does approach some fixed number as  $N$  increases. A thought experiment can illuminate the solution; the actual proof is the next paragraph. Consider the case of infinite players and, more importantly, infinite cards. Each card dealt to Alice has a  $\frac{12}{13}$  chance of not being a three. Each card is now virtually an independent event (not getting a three does not increase your chances of getting on later as in the finite case) hence the probability of no threes in a hand of thirteen cards is  $(\frac{12}{13})^{13}$ .

The last solutions shows us that the probability for  $N$  people is  $\prod_{i=0}^{12} \frac{12N-i}{13N-i}$ . We can rewrite this as  $\prod_{i=0}^{12} \frac{12-\frac{i}{N}}{13-\frac{i}{N}}$ . Therefore it is clear that each term of the product approaches  $\frac{12}{13}$  as  $N$  approaches infinity. Thus, the probability approaches  $(\frac{12}{13})^{13}$ .

*Designs*

17. Solutions may vary. One nice construction that yields an answer is the Fano Plane. There are 7 points and we want blocks of size 3 such that every pair appears once. Draw the seven points as below and consider the lines (and the one circle) as the blocks.  $B = \{(0, 1, 3), (0, 2, 6), (0, 4, 5), (1, 2, 4), (1, 5, 6), (2, 3, 5), (3, 4, 6)\}$ . Note that the Fano plane is the only  $(7,3,1)$  design up to equivalence.



18. Since each block is of size  $k$ ,  $\binom{k}{2}$  counts the total number of pairs in each block. As there are  $b$  blocks,  $b \cdot \binom{k}{2}$  is the number of all pairs of points that appear in a block together (with possible repetitions). Now, consider all possible pairs of points. There are  $\binom{v}{2}$  of these and each of these appear in the block  $\lambda$  times. Thus  $\lambda \binom{v}{2} = b \binom{k}{2}$  implying the result we desire.
19.  $r = \frac{\lambda(v-1)}{k-1}$ . We shall prove  $r(k-1) = \lambda(v-1)$  which implies the first equation. First, fix a point  $p \in P$ . There are  $r$  blocks containing  $p$  and each of these has  $k-1$  elements distinct from  $p$ .  $r \binom{k-1}{1}$  is the number of pairs of points in all blocks where one member of the pair is  $p$ . Now  $\binom{v-1}{1}$  is the number of possible pairs of  $p$  with some other element of  $P$ . Each possible pair appears  $\lambda$  times. Thus  $r(k-1) = \lambda(v-1)$ . Note that combining with our previous result yields  $bk = vr$ .
20. Any two distinct blocks have at most 1 point in common, since  $\lambda = 1$ . Choose a set  $S$  of 3 points not contained in any block together. For each pair of elements  $T$  in  $S$ , there is a unique block  $B_T$  containing the pair  $T$ . Each such  $B_T$  contains  $k-2$  points not in  $S$  and any point not in  $S$  is in at most one  $B_T$  since two such blocks already have a point of  $S$  in common. There are 3 possible different subsets of  $S$  of size 2, and thus there are 3 different  $B_T$ 's. So, this shows that the union of all blocks  $B_T$  contains  $3 + 3(k-2)$  distinct points and the result follows.