Geometry Solutions
2003 Rice Math Tournament
February 22, 2003

1. **Answer: 1**

   We may consider $\overline{AB}$ to be the base of $\triangle ABE$. Then, regardless of where point E is placed, the base of $\triangle ABE$ has length 1, and its height is 1, giving it an area of $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Hence, $m = M = \frac{1}{2}$. so $m/M = 1$.

2. **Answer: $C = (4, 5)$ or $C = (7, 8)$**

   If 3 points do not determine a unique triangle, then they must be collinear. Draw the line through A and B. If C is on the opposite side of B than A, then $AC = AB + BC = 3BC$, which implies that $BC = \frac{1}{2}AB$. If $m = B - A = (4, 4)$, then $C = B + \frac{1}{2}m = (7, 8)$.

   On the other hand, if C is between A and B, then $AC = 3BC$ implies that C is three fourths of the way from A to B. Thus, $C = A + \frac{3}{4}m = (4, 5)$.

   Finally, C cannot be on the opposite side of A because then $BC > AC$ which contradicts $3BC = AC$. Thus, the two possibilities for C that we have already found are the only two points it can be.

3. **Answer: 2 radians**

   Let R be the radius of the outer circle, and let r be the radius of the inner circle. Let θ be the measure of the arc that Selma walks. Then the distance Selma walks is $R\theta$, and the distance Patty walks is $2(R - r) + r\theta$. Setting them equal, we find that $R\theta = 2R - 2r + r\theta$, and $R(\theta - 2) = r(\theta - 2)$. Since $R > r$ by assumption, this is only possible if $\theta = 2$.

4. **Answer: $\sqrt{3} - \frac{\pi}{3}$**

   Draw two radii from O to the points of tangency along AB and AC, and also draw a line from O to A. This forms two congruent right triangles. Since each of the two triangles has sides of length 1, $\sqrt{3}$, and 2, it can be seen that $\text{m}\angle OAB = \text{m}\angle OAC = 30°$. Thus, $\triangle ABC$ is equilateral. So, area of the shaded region is one-third of the area remaining after we subtract the area of the triangle, and thus is $\frac{1}{3}(3\sqrt{3} - \pi) = \sqrt{3} - \frac{\pi}{3}$.

5. **Answer: $\frac{17}{2}$**

   Extend AD and BC to their intersection and call that E. Note that $\angle AEB$ is a right angle. The quantity we want is the difference between the medians to the hypotenuses of triangles ABE and CDE. Since the median to the hypotenuse of a right triangle is half the length of the hypotenuse, the answer is $\frac{1}{2}AB - \frac{1}{2}CD = \frac{17}{2}$.

6. **Answer: $2\sqrt{3} - 3$**

   Adding in the segment OC, we get an isosceles triangle OAC. Notice that $OA = OC = r$, the radius of the circle. Then, the law of cosines gives us $36 = r^2 + r^2 - 2r^2 \cos 120°$, which implies that $r = 2\sqrt{3}$.

   Now, observe that $\triangle AOE \sim \triangle CDE$. Thus,

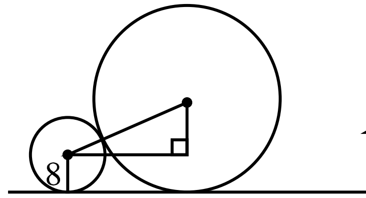
   $$\begin{aligned} \frac{AE}{EC} &= \frac{AO}{CD} \\ \frac{6 - EC}{EC} &= \frac{2\sqrt{3}}{\sqrt{3}}. \end{aligned}$$

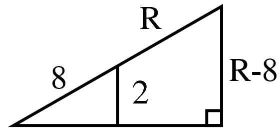
   Solving this equation, we find that $EC = 2$, and therefore $AE = 4$. Using the Pythagorean Theorem, we can then find $OE = 2$ and $ED = 1$. Thus

   $$DB = 2\sqrt{3} - 2 - 1 = 2\sqrt{3} - 3.$$

7. **Answer: $\frac{40}{3}$ inches**

   Since the point of contact is above the given ball's radius, then the other ball has a larger radius. Let R be the unknown radius. Draw a line from the center of one ball to the center of the other. The line drawn goes through the tangent point. We can make two similar right triangles as in the diagram below. Thus $\frac{R+8}{8} = \frac{R-8}{2}$, which implies $R = \frac{40}{3}$ inches.

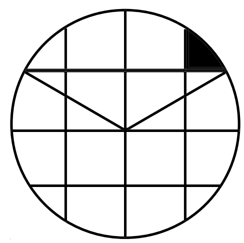




8. **Answer:** $\mathbf{\frac{16\pi}{3} + 16 - 16\sqrt{3}}$

The entire circle has an area of 64π, and the four squares in the center have total area 64. Call the area of a side piece s and the area of the corner piece c. Then $8s + 4c = 64\pi - 64$, which simplifies to $2s + c = 16\pi - 16$.

Now consider the triangle in the diagram below. The triangle has a $120°$ central angle, and its area is $\frac{1}{2} \cdot 8\sqrt{3} \cdot 4 = 16\sqrt{3}$. The area of the entire sector is $\frac{64\pi}{3}$, so the area of the sector minus the area of the triangle is $\frac{64\pi}{3} - 16\sqrt{3}$. However, the area of this region is also $2s + 2c$, which implies that $2s + 2c = \frac{64\pi}{3} - 16\sqrt{3}$. Subtracting the first equation from this equation yields $c = \frac{16\pi}{3} + 16 - 16\sqrt{3}$.



9. **Answer:** $\mathbf{2(\sqrt[4]{3})^3 \cdot (\frac{3}{2})^{n-1}}$

Let s be the length of a side in the first figure. Each angle is $60°$, and the area of the triangle is 1, so

$$\begin{aligned} \frac{1}{2}s^2 \sin 60° &= 1 \\ s^2 &= \frac{4}{\sqrt{3}} \\ s &= \frac{2}{\sqrt[4]{3}}. \end{aligned}$$

The perimeter of this figure is just $3s = 2(\sqrt[4]{3})^3$.

Now, notice that after each step, the perimeter of the shaded region increases by a factor of $\frac{3}{2}$. Thus, in the nth figure, the perimeter has increased by this factor $n-1$ times, so its perimeter is $2(\sqrt[4]{3})^3 \cdot (\frac{3}{2})^{n-1}$.

10. **Answer:** $\mathbf{\frac{\sqrt{3}+3}{8} - \frac{\pi}{96}[38 - 9\sqrt{2} + 2\sqrt{3} - 5\sqrt{6}]}$

First, we determine the area of $\triangle ABC$. We use the law of sines to find the length of the side opposite C in the triangle: $\frac{1}{\sin 45°} = \frac{c}{\sin 75°}$, so $c = \frac{\sqrt{3}+1}{2}$ (using the fact that $\sin 75° = \sin 45° \cos 30° + \cos 45° \sin 30°$). So, the area of the triangle is $\frac{1}{2} \cdot a \cdot c \cdot \sin B = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}+1}{2} \sin 60° = \frac{\sqrt{3}+3}{8}$.

Now, we determine the area that is inside the triangle of the three circles. First we find the length of the side opposite B in the triangle: $\frac{1}{\sin 45°} = \frac{b}{\sin 60°}$, so $b = \frac{\sqrt{3}}{\sqrt{2}}$. Now we need to find the radii of the circles. Let the circle with center A have radius x, the circle with center B have radius y, and the circle with center C have radius z. Then we know that $x + y = \frac{\sqrt{3}+1}{2}$, $x + z = \frac{\sqrt{3}}{\sqrt{3}}$, and $y + z = 1$. Solving for x, y, and z we determine that $x = \frac{\sqrt{3}+\sqrt{6}-1}{4}$, $y = \frac{3+\sqrt{3}-\sqrt{6}}{4}$, and $z = \frac{1+\sqrt{6}-\sqrt{3}}{4}$. Using these values, we can calculate the total area of the circles inside the triangle to be $\frac{\pi}{96}[38 - 9\sqrt{2} + 2\sqrt{3} - 5\sqrt{6}]$. So, the area of the triangle that is outside the circles is $\frac{\sqrt{3}+3}{8} - \frac{\pi}{96}[38 - 9\sqrt{2} + 2\sqrt{3} - 5\sqrt{6}]$.