Calculus Solutions
2003 Rice Math Tournament
February 22, 2003

1. **Answer: $\frac{100!}{x^{101}}$**

   Simply note the pattern: $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$.

2. **Answer: $\frac{34}{3}$**

   The distance traveled is given by $\int_0^4 |v(t)|dt$ where $|v(t)|$ is the speed function. $v(t) = \int a(t)dt = t^2 - 2t + C$. We can evaluate C with the information that $v(1) = -4$. $C = -3$. The distance covered is then $\int_0^4 |t^2 - 2t - 3|dt = \int_3^4 (t^2 - 2t - 3)dt - \int_0^3 (t^2 - 2t - 3)dt = \left[\frac{t^3}{3} - t^2 - 3t\right]_3^4 - \left[\frac{t^3}{3} - t^2 - 3t\right]_0^3 = \frac{7}{3} + 9 = \frac{34}{3}$.

3. **Answer: $\frac{\sqrt{5}}{3}$ or $-\frac{\sqrt{5}}{3}$ (one of these is sufficient)**

   The Mean Value Theorem says that there exists a c in the interval $[-1, 2]$ so that $f'(c) = \frac{f(2)-f(-1)}{2+1} = \frac{5}{3}$. Clearly, $f'(x) \neq \frac{5}{3}$ for $x > 1$. For $x \leq 1$, $f'(x) = 3x^2$ yielding $c = \pm\frac{\sqrt{5}}{3}$.

4. **Answer: -24**

   The only nonzero term in $f''(x)$ after substituting in $x = 3$ is the term in the product rule where we have differentiated the $(x - 3)$ terms twice. Thus $f''(x) = 2(3 - 1)(3 - 2)(3 - 4)(3 - 5)(3 - 6) = -24$. Note every term in $f'(x)$ involves $(x - 3)$ and thus $f'(3) = f(3) = 0$. Thus, $f''(3) - f'(3) + f(3) = -24$.

5. **Answer: $122.5 = \frac{245}{2}$**

   We know that $f(x + h) = f(x) + f(h) + 3xh$, so

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{f(x) + f(h) + 3xh - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h} + 3x \\ &= 3x + 7 \end{aligned}$$

   We note that this limit is exactly $f'(x)$, so we know $f'(x) = 3x + 7$ and thus $f(x) = 1.5x^2 + 7x$ (since $f(0) = 0$), and thus $f(7) = 122.5$.

6. **Answer: Yes, after ln 6 hours**

   Given that $c'(t) = -c(t)$, we can solve for $c(t)$ by separation of variables:

$$\begin{aligned} \frac{c'}{c} &= -1 \\ \ln|c| &= -t + k_1 \\ c &= e^{-t+k_1} \text{ (since } c \text{ is always positive)} \\ c &= k_2 e^{-t} \end{aligned}$$

   Plugging in $t = 0$, we find that $k_2 = c(0) = 2$, and therefore $c(t) = 2e^{-t}$.

   Now, given that $q'(t) = -3c(t)$, we find that

$$\begin{aligned} q' &= -6e^{-t} \\ q &= 6e^{-t} + k_3. \end{aligned}$$

   Plugging in $t = 0$, we see that $k_3 = q(0) - 6 = 1$, so $q(t) = 6e^{-t} + 1$. This function does indeed decrease below 2 as t increases, and this occurs when

$$\begin{aligned} 6e^{-t} + 1 &= 2 \\ e^{-t} &= \frac{1}{6} \\ -t &= \ln\frac{1}{6} \\ t &= \ln 6. \end{aligned}$$

   Hence, Teena gets kicked out after $\ln 6$ hours.

7. **Answer:** $\boldsymbol{(x^x \ln x)(x^{x^x})(\ln x + 1 + \frac{1}{x\ln x}) = x^{x^{(x+1)}} \ln x(\ln x + 1 + \frac{1}{x\ln x}) = x^{x^{(x+1)}}((\ln x)^2 + \ln x + \frac{1}{x})}$

   Set $y = x^{x^x}$. Then $\ln y = x^x \ln x$ and $\ln\ln y = \ln x^x \ln x = x\ln x + \ln\ln x$. Taking the derivative implicitly yields $\frac{1}{\ln y}\cdot\frac{1}{y}\cdot y' = \ln x + 1 + \frac{1}{\ln x}\cdot\frac{1}{x}$. Thus $y' = (x^x \ln x)(x^{x^x})(\ln x + 1 + \frac{1}{x\ln x}) = x^{x^{(x+1)}} \ln x(\ln x + 1 + \frac{1}{x\ln x}) = x^{x^{(x+1)}}((\ln x)^2 + \ln x + \frac{1}{x})$.

8. **Answer:** $\boldsymbol{(\sqrt{s} - \sqrt{x})^2}$

   Consider a fixed x_0 with $0 \le x_0 \le s$. Consider a line going from $(0, r)$ to $(s-r, 0)$ (where $0 \le s-r \le x_0$). The y-value of this line at x_0 is $r \cdot \left(1 - \frac{x_0}{s-r}\right)$. So, $f(x_0)$ will be equal to $\max_{0\le r\le s-x_0} r \cdot \left(1 - \frac{x_0}{s-r}\right)$. We let $g(r) = r \cdot \left(1 - \frac{x_0}{s-r}\right)$ and find the maximum of this function by setting the derivative equal to 0:

$$\begin{aligned} g'(r) &= \left(1 - \frac{x_0}{s-r}\right) + r \cdot \left(-\frac{x_0}{(s-r)^2}\right) \\ &= 1 - \frac{x_0}{s-r} - \frac{rx_0}{(s-r)^2} \end{aligned}$$

   Setting this equal to 0, we see:

$$\begin{aligned} 1 - \frac{x_0}{s-r} - \frac{rx_0}{(s-r)^2} &= 0 \\ (s-r)^2 - x_0(s-r) - rx_0 &= 0 \\ r^2 - 2rs + s(s-x_0) &= 0 \end{aligned}$$

   Solving for r, we find that $r = s \pm \sqrt{sx_0}$ is a solution, and since $s + \sqrt{sx_0} > s$, which is not allowed, then $r = s - \sqrt{sx_0}$ must be the maximum, which we can check by finding the second derivative.

   So, we have $f(x_0) = g(s - \sqrt{sx_0}) = (s - \sqrt{sx_0})\left(1 - \frac{x_0}{\sqrt{sx_0}}\right)$, which reduces to $f(x_0) = (\sqrt{s} - \sqrt{x_0})^2$, and thus $f(x) = (\sqrt{s} - \sqrt{x})^2$.

9. **Answer:** $\boldsymbol{\frac{\sqrt[3]{5}}{90}}$ **inches per sec.**

   The total volume of the hourglass is 480π and thus the volume of the sand is 120π. The $\frac{dv}{dt}$ of the sand is a constant, thus after $\frac{3}{4}$ of a minute, $90\pi\text{in}^3$ of sand is in the bottom cone. Note $\frac{dv}{dt} = 2\pi\frac{\text{in}^3}{\text{sec}}$. If h is the height of the sand in the bottom cone and r the radius of the top surface of the sand, then by similar triangles $h = 5 - \frac{5r}{12}$, and the volume of the sand in the bottom is $V = 240\pi - \frac{1}{3}\pi r^2(5-h) = 240\pi - \frac{5\pi r^3}{36}$. Setting $\frac{dv}{dt} = 2\pi$ yields $\frac{dr}{dt} = \frac{-24}{5r^2}$, giving that $\frac{dh}{dt} = \frac{2}{r^2}$. We can solve for r when $V = 90\pi$ giving $r = 6\sqrt[3]{5}$. Thus $\frac{dh}{dt} = \frac{\sqrt[3]{5}\text{ in}}{90\text{ sec}}$.

10. **Answer:** $\boldsymbol{\frac{5456\pi}{3} + 198\sqrt{11}\pi - \frac{286\sqrt{143}\pi}{3}}$ **cubic inches**

    First, Sammy removes $\frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$ cubic inches from the center of the pumpkin. Next his drill removes a volume that is almost a cylinder, but is curved on the ends. To calculate these small volume changes we shall derive a general formula. Consider a circle of radius r and the region bounded by $x^2 + y^2 = r^2$, $x = a$, and $y = 0$. If we rotate this around the x-axis, we get the extra volume on the outer edge of the cylinder. Thus the volume is $\pi\int_a^r (r^2 - x^2)dx = \pi(\frac{2}{3}r^3 - ar^2 + \frac{a^3}{3})$. For the outside of the cylinder, $r = 12$ and $a = \sqrt{r^2 - 1}$. We are thus removing from the pumpkin an additional volume of $\pi(1152 - \frac{289\sqrt{143}}{3})$. On the inner radius we are overestimating what volume we are removing by $\pi(\frac{2000}{3} - 201\sqrt{11})$. Our initial estimate of the cylinder was $\pi(\sqrt{143} - \sqrt{99})$. Adding these numbers yields the solution of $\frac{5456\pi}{3} + 198\sqrt{11}\pi - \frac{286\sqrt{143}\pi}{3}$ cubic inches.