Algebra Solutions
2003 Rice Math Tournament
February 22, 2003

1. **Answer: -3**

   If $a * b = 2$, then $a^2 + ab + 3b + 1 = 2$, so $b(a + 3) = 1 - a^2$. From this we see that if $a \neq -3$, then $b = (1-a^2)/(a+3)$ satisfies $a*b$. However, if $a = -3$, then for any b, $a*b = a^2+(a+3)b+1 = a^2+1 = 10$.

2. **Answer: 4**

   a and b are the roots of the polynomial $x^2 - kx + k$. (There are various ways to show this. One is to multiply out $(x - a)(x - b)$, another is to substitute $b = k - a$ into the equation $ab = k$.) Using the quadratic equation, this polynomial has roots

   $$x = \frac{k \pm \sqrt{k^2 - 4k}}{2}.$$

   These roots are real if and only if $k^2 - 4k \geq 0$. Since $k > 0$, we find that this only holds if $k \geq 4$.

3. **Answer: 4**

   Since $p(x)$ has no complex roots, all 8 of its roots are real. Note however that the polynomial is even, i.e. symmetric about the y-axis. Thus, $P(x)$ has an equal number of positive and negative roots, meaning that it has $\frac{8}{2} = 4$ negative real roots.

4. **Answer: 1 and 8**

   Let A, B, and C be the number of galleons Harry, Hermione, and Ron have, respectively. Then the given information provides us with the following conditions:

   $$\begin{aligned} A + B &= 12 \\ A + C &= 10 \\ A + B + C &= 7D, \text{ where } D \text{ is an integer} \end{aligned}$$

   By adding the first two and subtracting the third, we get $A = 22 - 7D$. Since A, B, and C must be nonnegative, we have $A = 1$ or 8.

5. **Answer: $(1, 2)$**

   Written in base 10, the equation $(11xy)_7 = (310x)_5$ becomes

   $$\begin{aligned} 7^3 + 7^2 + x \cdot 7^1 + y &= 3 \cdot 5^3 + 5^2 + x \\ 392 + 7x + y &= 400 + x \\ 6x + y &= 8. \end{aligned}$$

   Since x is a digit in a base 5 number, we must have $0 \leq x \leq 4$, and since y is a digit in a base 7 number, we must have $0 \leq y \leq 6$. The only solution that satisfies these constraints is $x = 1$, $y = 2$.

6. **Answer: 4446**

   Let w be the number of white and r be the number of red. Then $r + w < 100$ and and $w < 10r$. We want the number of integer coordinates (r, w) that satisfy these two inequalities. Consider the lines $r+w = 100$ and $\frac{w}{r} = 10$. They intersect at $(\frac{100}{11}, \frac{1000}{11})$. To be in the desired region for $r < \frac{100}{11} \approx 9.1$, we are limited more by $w < 10r$. For $r = 1$ there are 9 choices for w. For $r = 2$ there are 19 choices, and so on up to $r = 9$ which has 89 choices for w. Therefore for $r \leq 9$, there are $\sum_{n=1}^{9}(10n - 1) = 441$. When $r \geq 10$, then w is more restricted by $w < 100-r$. When $r = 10$, there are 89 choices for w. When $r = 11$ there are 88 choices and so on. Thus, the number of choices when $r \geq 10$ is $\sum_{n=1}^{89} n = \frac{89 \cdot 90}{2} = 4005$. The total number of ordered pairs is $4005 + 441 = 4446$.

7. **Answer: 14**

   Since r_1, r_2, and r_3 are solutions, we know that

   $$(x - r_1)(x - r_2)(x - r_3) = x^3 - 2x^2 + 4x + 10.$$

Plugging in $x = -2$ gives us

$$\begin{aligned} (-2 - r_1)(-2 - r_2)(-2 - r_3) &= (-2)^3 - 2(-2)^2 + 4(-2) + 10 \\ (-1)(r_1 + 2)(r_2 + 2)(r_3 + 2) &= -14 \\ (r_1 + 2)(r_2 + 2)(r_3 + 2) &= 14. \end{aligned}$$

8. **Answer: $\boldsymbol{x = 16}$**

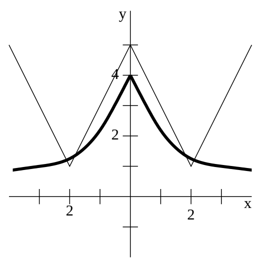
   First, we want to know the relationship between $\log_2 a$ and $\log_4 a$ for any positive number a. Let $b = \log_4 a$. Then $a = 4^b = 2^{2b}$, which implies that $\log_2 a = 2b$. Therefore, $\log_2 a = 2 \log_4 a$.

   Now, let $y = \log_4 x = \frac{\log_2 x}{2}$. Substituting this into the equation for x, we find that $\log_2 y + \log_4 2y = 2$. Noting that $\log_2 y = 2 \log_4 y = \log_4 y^2$, we then obtain

$$\begin{aligned} \log_4 y^2 + \log_4 2y &= 2 \\ \log_4 2y \cdot y^2 &= 2 \\ 2y^3 &= 4^2 \\ y &= 2. \end{aligned}$$

   And finally, $x = 4^y = 16$.

9. **Answer: $[\frac{3+\sqrt{17}}{4}, \frac{1+\sqrt{57}}{4}], [\frac{-1-\sqrt{57}}{4}, \frac{-3-\sqrt{17}}{4}]$ or $\frac{3+\sqrt{17}}{4} \leq |x| \leq \frac{1+\sqrt{57}}{4}$**



   Graph each of the functions. The darker curve is the left side of the inequality. We can see the graph is symmetric about the y-axis, so we may just sovle for when $x > 0$. Thus $|x| = x$. Solving for equality first, we have that $4 = (1 + |2x - 4|)(x + 1)$. First consider when $2x - 4 \geq 0$. In that case we are solving $2x^2 - x - 7 = 0$ which has roots $x = \frac{1 \pm \sqrt{57}}{4}$. Remember we assumed $x > 0$ so we want only the positive root. To find the other positive intersection point consider when $2x - 4 < 0$. In that case, $4 = (1 + |2x - 4|)(x + 1)$ is equivalent to $2x^2 - 3x - 1 = 0$, and the root we want is $x = \frac{3+\sqrt{17}}{4}$. Thus the solutions are $[\frac{3+\sqrt{17}}{4}, \frac{1+\sqrt{57}}{4}], [\frac{-1-\sqrt{57}}{4}, \frac{-3-\sqrt{17}}{4}]$ or $\frac{3+\sqrt{17}}{4} \leq |x| \leq \frac{1+\sqrt{57}}{4}$.

10. **Answer: $\frac{b+\sqrt{b^2+4c}}{2}$**

    For each positive integer n, let $r_n = x_{n+1}/x_n$. Taking the recurrence relation for x_{n+2} and dividing by x_{n+1}, we get that $\frac{x_{n+2}}{x_{n+1}} = b + c\frac{x_n}{x_{n+1}}$ and $r_{n+1} = b + cr_n^{-1}$, which is a recurrence relation for r_n.

    Based on the given information, we know that the sequence $r_1, r_2, r_3, \ldots$ approaches R as n goes to ∞. Then R must be a fixed point of the recursion for r_n; that is, it must satisfy

$$R = b + cR^{-1}.$$

    (If R *didn't* satisfy this relation, then whenever the sequence r_n got very close to R, it would immediately jump away from R.) Therefore

$$R^2 - bR - c = 0.$$

    Using the quadratic equation, we find that

$$R = \frac{b \pm \sqrt{b^2 + 4c}}{2}.$$

    One can easily see that x_n is positive for all n, and therefore r_n must be positive for all n as well. Therefore, R must be positive, too, meaning that the only possibility is

$$R = \frac{b + \sqrt{b^2 + 4c}}{2}.$$