Advanced Topics Test
2003 Rice Math Tournament
February 22, 2003

1. Let $\diamondsuit$ be a binary operator on positive real numbers that satisfies the following two rules: $(x \cdot y^2) \diamondsuit y = x(y \diamondsuit 1)$ and $(x \diamondsuit 1) \diamondsuit x = 1$. Given $1 \diamondsuit 1 = 1$, find $23 \diamondsuit 87$.

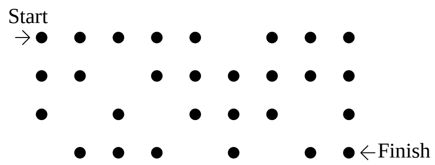
2. Two coins are found in a fountain. One is a fair coin and the other has "heads" on both sides. One coin is chosen randomly and flipped 10 times. All 10 times it lands "heads" face up. What is the probability that the fair coin was chosen?

3. Sammy and Bevo each choose a real number at random between 1 and 10, inclusive. What is the probability that they differ by more than 4?

4. We have polynomials $a(x), b(x), c(x), d(x)$, none of which have roots of multiplicity > 1, and we know that $\log_2 a(x) = \log_2 b(x) + \log_2 c(x) - \log_2 d(x)$. We know that $b(x) = 0$ has exactly two solutions, $x = -7$ and $x = -11$, $c(x) = 0$ has exactly two solutions, $x = -5$ and $x = -8$, and $d(x) = 0$ has exactly two solutions, $x = -5$ and $x = -11$. Also, $\log_2 a(-6) = 2$. Find $\log_2 a(1)$.

5. There are 1,000 points equally spaced on a circle of radius 10. Six points are chosen randomly; call them A, B, C, D, E and F (in some order). What is the probability that the triangles ADC and BEF do not intersect each other?

6. In the diagram below, how many distinct paths are there from the starting dot to the ending dot? You can move from one dot to another adjacent dot by moving right, down, diagonally down to the right or diagonally up to the right, and two dots are adjacent if they are within one row and/or one column of each other.



7. All thirteen spades in a deck of cards are shuffled uniformly and dealt in a line. Let S be a statement about the order of the thirteen cards and $P(S)$ be the probability that S is true. For example, suppose S is "The five appears before the nine", then $P(S) = \frac{1}{2}$. How many of the values $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{50}$ can $P(S)$ not attain?

8. Euler and his wife (Katharina Gsell) throw a dinner party and invite four other married couples. Once everyone arrives, various people shake hands. Note that no person shakes hands with himself and no married couple shakes hands with each other. Euler asks his wife and everyone else at the party how many people's hands they have shaken and is shocked to find that every answer he receives is different (note that Euler doesn't consider the number of handshakes in which he participated). How many handshakes did Euler's wife participate in?

9. Bored of playing football, a group of mathematicians devise an untimed game called Boppo. The players ride around on brooms and bop each other with Nerf™ Klein bottles. If a player bops another player when the last digit of the seconds on the official clock is 9, then he scores 9 points. If it is a 7, he scores 10 points. In the off chance that the clock's seconds digits are the same (00, 11, 22, … or 55), then the player scores 17 points. If none of these apply, he gets no points. What is the product of the two greatest scores that a player cannot score in the game?

10. Evaluate $2 \cdot 3^{-1} - 4 \cdot 3^{-2} + 6 \cdot 3^{-3} - 8 \cdot 3^{-4} + \cdots$.