Strategyproof profit sharing: a two-agent characterization

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Abstract

Two agents jointly operate a decreasing marginal returns technology to produce a private good. We characterize the class of output-sharing rules for which the labor-supply game has a unique Nash equilibrium. It consists of two families: rules of the *serial* type which protect a small user from the negative externality imposed by a large user, and rules of the *reverse serial* type, where one agent effectively employs the other agent's labor. Exactly two rules satisfy symmetry; a result in sharp contrast with Moulin and Shenker's (*Econometrica*, 1992) characterization of their serial mechanism as the unique *cost*-sharing rule satisfying the same incentives property. We also show that the familiar *stand alone test* characterizes the class of *fixed-path methods* (Friedman, *Economic Theory,* 2002) under our incentives criterion.

Keywords: Joint production, serial rule, decreasing serial rule, strategyproofness.

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1 Introduction

When several producers jointly operate a production process, total output (or profits) must be shared as a function of their individual contributions (see Israelsen 1980, Sen 1966, Weitzman 1974). This question applies whether the production structure is one of common access to the production function (as in the so-called "commons problem") or one where property rights to the technology are clearly defined. An extreme example of the latter is that of a monopolist hiring workers.

The production possibilities are common knowledge and exhibit decreasing marginal returns, but the individual leisure-consumption trade-offs are private information. We are concerned with sharing rules with very strong incentives properties, so as to avoid undesirable phenomena like free-riding or the familiar "tragedy of the commons". The incentives criterion we consider is that of *strategyproofness (SP)*, under which it is a dominant strategy for every agent to behave according to her true preference. This requirement does not hinge on any informational assumption and is therefore more robust than, say, Bayesian incentive compatibility.[1]

In the two-agent case, we characterize the class of sharing rules which are monotonic (each agent's share is increasing in her own input contribution), smooth (the sharing rule is continuously differentiable in inputs) and which satisfy an incentives requirement even stronger than SP (Theorem 1). This class of sharing rules is made up of two families which we call the "serial" family and the "reverse serial" family. An essential feature of rules in the serial family is that the share of a relatively small supplier of input is unaffected by changes in the supply level of a large supplier (a feature called the "serial principle" in Sprumont, 1998) while the converse is true for rules of the reverse serial type: the share of a large supplier is unaffected by changes in the input level of a small supplier.

In addition, the externality imposed by a small user on large users is negative under a serial rule. Conversely, a large supplier of input imposes positive externalities on small suppliers under a reverse serial rule. Thus, we argue that serial rules are more adapted to the commons problem, with the negative externality reflecting congestion, while reverse serial rules correspond to a more "corporate" production structure where the owner of the facility extracts rents from the labor contribution of a worker.

In Section 5 we consider a popular axiom in the commons literature. The *stand alone test (SA)* captures the essence of the commons problem by demanding that no agent be made better off by the presence of others than if she were operating the technology by herself (see, e.g., Moulin and Shenker 1992, Suh 1997, Sprumont 1998, Hougaard and Thorlund-Pertersen 2000). It turns out that SA characterizes the output-sharing version of the class of *fixed-path methods (FPMs)* discussed in Friedman (2002, 2004) (see Theorem 2). These sharing rules allocate marginal quantities of input, and the corresponding amounts of output, along a prespecified path in the agents' input space. Among well-known FPMs are the Moulin and Shenker serial rule and priority rules, which follow the diagonal of the positive orthant and an axis of the agents' input space, respectively.

After relating our work to the existing literature (Section 2) we define the serial and reverse serial families of sharing rules (Section 3) and state our main characterization theorem (Theorem 1) in Section 4. In Section 5 we characterize the class of FPMs by SA.

[1] We refer the reader to a companion paper (Leroux, 2005) for a discussion of strategyproofness relative to Bayesian incentive compatibility.

2 Relation to the literature

This work contributes to the large literature on the trade-off between efficiency and incentive compatibility.

Because mechanisms in the serial and reverse serial family are strategy-proof, they fail to be first-best efficient (see Leroux, 2004). The bulk of the recent literature on strategyproofness in the cooperative production of a private good was framed in the cost-sharing context, where agents demand a quantity of output and split the cost of meeting total demand (see Moulin and Shenker 1992, Shenker 1992, de Frutos 1998, Sprumont 1998, Friedman 2002, 2004, Téjédo and Truchon 2002, and Alcalde and Angel-Silva 2004). By contrast, Suh (1997) is the only other contribution that we know of which adopts the output-sharing framework. We provide a discussion of the subtle difference between the cost- and output-sharing frameworks in Section 6. We refer the reader to Moulin (2002) for a survey on axiomatic cost and output sharing.

Regardless of the framework adopted, mapping out the class of strategy-proof mechanisms in economies with production of private goods remains a large open question. So far, authors have mainly approached the question by pairing SP with additional axioms: e.g. symmetry (Moulin and Shenker 1992, Suh 1997), or individual rationality (Leroux, 2005). Our Theorem 1 is the first to be free of such axioms.

The FPMs we characterize in Theorem 2 are the output-sharing versions of cost-sharing mechanisms introduced Friedman (2002) as non-anonymous generalizations of the Moulin and Shenker serial rule retaining its strong incentives properties. We show in a companion paper (Leroux, 2005) that Theorem 2 does not extend to the many-agent case and discuss the appeal of FPMs in partnership problems.

Recent related literature on the common production of private goods considers weaker interpretations of incentive compatibility (see, e.g., Corchón and Puy 2002, Shin and Suh 1997). For instance, Corchón and Puy establish that any continuous sharing rule admits a Pareto-efficient allocation which can be Nash-implemented. Yet, any game implementing such an outcome must have several, non-welfare-equivalent Nash equilibria at some profiles. Here we insist on the uniqueness of the Nash equilibrium, a much more demanding requirement than the above kind of Nash-implementability.

3 The two families

Two agents jointly operate an increasing, strictly concave and continuously differentiable production function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that $F(0) = 0$. When each agent i supplies $x_i \geq 0$ units of input, the input vector $x = (x_1, x_2)$ yields $F(x_1 + x_2)$ units of total output.

Agent i's utility from supplying $x_i \geq 0$ and receiving $y_i \geq 0$ units of output is $u_i(x_i, y_i)$; the *utility function*, u_i, is decreasing in x_i, increasing in y_i and quasi-concave. A *preference profile* (or a *profile*) is a pair (u_1, u_2) of utility functions, one per agent.

A *sharing rule* is a mapping $\xi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that $\xi_1(x) + \xi_2(x) = F(x_1 + x_2)$ for all x, which is *smooth* (ξ is continuously differentiable) and *monotonic* ($\frac{\partial \xi_i}{\partial x_i} > 0$ for $i = 1, 2$).

We denote by $\mathcal{B}$ the class of non-decreasing functions $b : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ which are continuous on $\mathbb{R}_+$ and increasing on $\{t \geq 0 | 0 < b(t) < +\infty\}$. I.e., b can only be constant on a range where it returns zero; also, its graph can have a vertical asymptote. We denote by $\mathcal{F}$ the class of mappings from $\mathbb{R}_+$ to itself which are increasing, strictly concave and continuously differentiable.

Definition 1 *A sharing rule ξ is of the* serial type *if there exists $b \in \mathcal{B}$ and $g^s, h^s \in \mathcal{F}$ s.t.*

$$\xi(x) = \begin{cases} (g^s(x_1), F(x_1+x_2) - g^s(x_1)) & \text{if } x_2 \geq b(x_1) \\ (F(x_1+x_2) - h^s(x_2), h^s(x_2)) & \text{if } x_2 \leq b(x_1) \end{cases}$$

We denote by $\mathcal{S}$ the class of such rules.

The reader will notice that when $x_2 \geq b(x_1)$ agent 1's output share is unaffected by changes in x_2 above $b(x_1)$; a symmetric statement holds if $x_2 \leq b(x_1)$. Rules of the serial type protect low-level users of the facility from the negative externalities imposed by high-level users above a certain level.

It is easily checked that $\mathcal{S}$ includes the output-sharing version of the Moulin and Shenker serial rule (*the* serial rule, see Moulin and Shenker 1992) with $b(x_1) = x_1$, $g^s(t) = h^s(t) = \frac{1}{2}F(2t)$; and of the fixed-path methods discussed in Friedman (2002, 2004): with $g^s(x_1) = \int_0^{x_1} F'(t + b(t))dt$ and

$$h^s(x_2) = \begin{cases} F(x_2) & \text{if } x_2 \leq b(0), \\ F(b(0)) + \int_{b(0)}^{x_2} F'(b^{-1}(t) + t)dt & \text{otherwise.} \end{cases}$$

Definition 2 *A sharing rule ξ is of the* reverse serial type *if there exists $b \in \mathcal{B}$ and $g^r, h^r \in \mathcal{F}$ s.t.*

$$\xi(x) = \begin{cases} (F(x_1+x_2) - h^r(x_2), h^r(x_2)) & \text{if } x_2 \geq b(x_1) \\ (g^r(x_1), F(x_1+x_2) - g^r(x_1)) & \text{if } x_2 \leq b(x_1) \end{cases}$$

We denote by $\mathcal{R}$ the class of such mechanisms.

Here, however, agent 2's output level is unaffected by changes in x_1 *below* $b^{-1}(x_2)$ if $x_2 \geq b(x_1)$.[2] If $x_2 \geq b(x_1)$, a mechanism of the reverse serial type provides a compensation schedule, h^r, for the high-level supplier of input (agent 2, "the worker") whose labor benefits the low-level supplier (agent 1, "the employer").

Clearly, the *decreasing serial rule* (as in de Frutos [6]) belongs to $\mathcal{R}$: $b(x_1) = x_1$, $g^r(t) = h^r(t) = \frac{1}{2}F(2t)$.

The intersection of $\mathcal{R}$ and $\mathcal{S}$ is nonempty. It is worth noting that the priority rules giving full access to one agent belong to both families: for instance, both [$b \equiv 0$; $g^s(x_1) = F(x_1)$] and [$b \equiv +\infty$; $g^r(x_1) = F(x_1)$] represent the rule giving priority to agent 1. More generally we denote by $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) the class of rules where $b \equiv 0$ (resp. $b \equiv +\infty$), such that agent 1 (resp. 2) is a *dictator*, and by $\mathcal{D} \equiv \mathcal{D}_1 \cup \mathcal{D}_2$ the class of *dictatorships*. The reader can easily check that $\mathcal{R} \cap \mathcal{S} = \mathcal{D}$.

4 Main result

Our main result is a full characterization of the class of sharing rules satisfying an incentives criterion stronger than SP.

Theorem 1 *Let ξ be a sharing rule. The following statements are equivalent:*

[2] Note that from the definition of $\mathcal{B}$, $b^{-1}(x_2)$ exists for any positive x_2 in the range of b.

i) $\xi \in \mathcal{S} \cup \mathcal{R}$,

ii) the supply game associated with ξ *(strategy* x_i*, payoff* $u_i(x_i, \xi_i(x))$*) admits at most one Nash equilibrium at all profiles,*

iii) the supply game associated with ξ *admits exactly one Nash equilibrium at all profiles.*

A standard result in the implementation literature (See Dasgupta et al., 1979) implies that the direct mechanism—where each agent's strategy space is the space of utility functions—associated with a sharing rule $\xi \in \mathcal{S} \cup \mathcal{R}$ is SP. Moreover, the unique Nash equilibrium of the supply game turns out to be strong (easily checked), therefore the associated direct mechanism is also *group-strategyproof* (i.e. invulnerable to coordinated manipulations).

5 The stand alone test

Given that F exhibits decreasing marginal returns, the context is one of negative externalities where the participation of each agent decreases the productivity of the others. In a commons problem the mechanism designer may require that the sharing rule reflects these negative externalities. We propose the following interpretation of this requirement.

The stand alone test (SA) *A sharing rule,* ξ*, satisfies* SA *if and only if*

$$\xi_i(x) \leq F(x_i)$$

for all $x \in \mathbb{R}^2_+$ *and* $i = 1, 2$.

SA asks that no agent benefits from the presence of the other agent. We show that it characterizes the output-sharing versions of the class of fixed-path methods (Friedman, 2002) among rules of the serial and reverse serial family. Fixed-path methods allocate marginal increments of input—and the corresponding amount of output—along a prespecified continuous increasing path in the agents' input space. With our notations, the class of fixed-path methods consists of the two priority rules as well as all the non-dictatorial serial rules with $g^s(0) = h^s(0) = 0$:

$$FPM = \{\xi \in \mathcal{S} \backslash \mathcal{D} | g^s(0) = h^s(0) = 0\} \cup \{P^1, P^2\}$$

where $P^i_i(x) = F(x_i)$ and $P^i_j = F(x_i + x_j) - F(x_i)$.

Theorem 2 *Let* $\xi \in \mathcal{S} \cup \mathcal{R}$*, the following statements are equivalent:*

i) $\xi \in FPM$*,*

ii) ξ *satisfies SA.*

Proof. *Notation:* Define $X_1 = \{x_1 > 0 | 0 < b(x_1) < +\infty\}$ and write $X_1 =]\underline{x}_1, \bar{x}_1[$. Notice that $X_1 \neq \emptyset$ if and only if $\xi \notin \mathcal{D}$.

i) $\Longrightarrow$ **ii)** Suppose $\xi \in FPM$. Consider the following property:

Cross Monotonicity (CM) *A sharing rule,* ξ*, satisfies* CM *if and only if*

$$\frac{\partial \xi_i}{\partial x_j} \leq 0$$

on $\mathbb{R}^2_+$ *for* $i \neq j$.

Because CM is more demanding than SA, it suffices to show that ξ satisfies CM.[3] If $\xi = P^1$, then clearly $\frac{\partial \xi_1}{\partial x_2} \equiv 0$ and $\frac{\partial \xi_2}{\partial x_1}(x) = F'(x_1 + x_2) - F'(x_1) \leq 0$ for all x by the concavity of F. Similarly, P^2 also satisfies CM.

If $\xi \in \mathcal{S} \backslash \mathcal{D}$, then smoothness implies:

$$F'(x_1 + b(x_1)) = g^{s\prime}(x_1) = h^{s\prime}(b(x_1))$$

for all $x_1 \in X_1$. The reader can check that integrating between $(0,0)$ and (x_1, x_2) and taking into account $g(0) = h(0) = 0$ yields $g^s(x_1) = \int_0^{x_1} F'(t + b(t))dt$ and

$$h^s(x_2) = \begin{cases} F(x_2) & \text{if } x_2 \leq b(0), \\ F(b(0)) + \int_{b(0)}^{x_2} F'(b^{-1}(t) + t)dt & \text{otherwise.} \end{cases}$$

CM follows from the strict concavity of F and the strict monotonicity of b.

ii) $\Longrightarrow$ **i)** Suppose $\xi \in \mathcal{D}_1$, i.e. $\xi(x) = (g(x_1), F(x_1 + x_2) - g(x_1))$ for some $g \in \mathcal{F}$. By SA, $g(x_1) \leq F(x_1)$ and $\xi_2(x_1, 0) = F(x_1) - g(x_1) \leq F(0) = 0$ for all $x_1 \geq 0$. Hence, $g \equiv F$ and $\xi = P^1$. Similarly, if $\xi \in \mathcal{D}_2$ then $\xi = P^2$.

Suppose $\xi \notin \mathcal{D}$, we show that $\xi \notin \mathcal{R} \backslash \mathcal{D}$. By contradiction, suppose $\xi \in \mathcal{R} \backslash \mathcal{D}$, then for any $x_1 \in X_1$,

$$\begin{array}{lll} & \xi_2(x) = F(x_1 + x_2) - g^r(x_1) \leq F(x_2) & \text{for any } x_2 \leq b(x_1) \text{ by SA,} \\ \Longleftrightarrow & g^r(x_1) \geq F(x_1 + x_2) - F(x_1) & \text{for any } x_2 \leq b(x_1), \\ \Longleftrightarrow & g^r(x_1) \geq F(x_1) & \text{by concavity of } F, \\ \Longleftrightarrow & g^r \equiv F \text{ on } X_1, & \text{by SA (for agent 1).} \end{array}$$

Also, by smoothness

$$F'(x_1 + b(x_1)) = g^{r\prime}(x_1)$$

for all $x_1 \in X_1$ which is incompatible with both $g^r \equiv F$ and the strict concavity of F.

Finally, we show that if $\xi \in \mathcal{S} \backslash \mathcal{D}$, then SA implies $g^s(0) = h^s(0) = 0$. Indeed, if $0 < b(0) < +\infty$, then $\xi_2(0,0) = h^s(0) = 0$ and $\xi_1(0, b(0)) = g^s(0) = 0$. Similarly, the result holds if $\underline{x_1} > 0$ and if $b^{-1}(\{0\}) = \{0\}$. ■

6 Concluding comment

The following remark is an immediate corollary of Theorem 1 which generalizes to the many-agent case (see Suh 1997).

Remark 3 *The Moulin and Shenker serial rule and the de Frutos' decreasing serial rule are the only two symmetric sharing rules satisfying our incentives requirement (proviso* iii) *of the statement of Theorem 1).*

[3] Under CM, budget balance and the positivity of output shares require:

$$F(x_i) \geq \xi_i(x_i, 0) \geq \xi_i(x) \qquad \text{for all } x \text{ and all } i.$$

As recalled in Section 2, most of the existing literature on cooperative production focuses on the cost-sharing approach, with the general intuition that the output-sharing problem is a mere rewriting of the cost-sharing problem (see, e.g., Section 8 of Moulin and Shenker, 1992). However, the following contrasts markedly with this view.

Define the cost function $C \equiv F^{-1}$; clearly, C and F are equivalent representations of the same production possibilities. A *cost-sharing rule*, ζ, allocates to any vector of demands $(y_1, y_2) \in \mathbb{R}_+^2$ a cost vector $(x_1, x_2) \in \mathbb{R}_+^2$ such that $x_1 + x_2 = C(y_1 + y_2)$. The Moulin and Shenker serial rule, defined by

$$\zeta_i^{MS}(y) = \frac{1}{2}C(2y_i) \quad \text{and} \quad \zeta_j^{MS}(y) = C(y_1 + y_2) - \frac{1}{2}C(2y_i)$$

if $y_i \leq y_j$, is the only cost-sharing rule whose associated demand game has a unique Nash equilibrium at all profiles (Theorem 2 in Moulin and Shenker, 1992). In particular, the de Frutos decreasing serial cost-sharing rule, defined by

$$\zeta_i^{dF}(y) = \frac{1}{2}C(2y_j) \quad \text{and} \quad \zeta_j^{dF}(y) = C(y_1 + y_2) - \frac{1}{2}C(2y_j)$$

if $y_i \leq y_j$, is not well defined when C is strictly convex (i.e. when F is strictly concave, as in our framework) as it does not guarantee positive cost shares. This fact, in light of Remark 3, suggests that the difference between the cost- and output-sharing versions of the sharing problem goes beyond simple rewriting.

This discrepancy between these two formulations of the same problem is reminiscent of a somewhat different finding in Moulin and Watts (1997). They show that if given the choice between playing the average cost game or the average returns game, individuals would unambiguously choose the latter. We deem worthy of exploration whether adopting the cost- or output-sharing framework has other practical consequences.

A Appendix

A.1 Proof of Theorem 1

i) $\Longrightarrow$ *iii)* The proofs of the strategic properties of the Moulin and Shenker serial rule (See Moulin and Shenker, 1992, Theorem 1) and of fixed-path methods (see Friedman 2002) in the cost-sharing context can be adapted to our setting. We nonetheless provide a proof for the sake of completeness.

We will use the following lemma extensively; its obvious proof is omitted.

Lemma 1 *Suppose $f, g \in \mathcal{F}$ coincide on the interval $\left[\lambda^-, \lambda^+\right]$, and for any utility function, u, denote $\lambda_1 = \max_\lambda u(\lambda, f(\lambda))$ and $\lambda_2 = \max_\lambda u(\lambda, g(\lambda))$. The following statements are true:*

- $\lambda_1 < \lambda^-$ *iff* $\lambda_2 < \lambda^-$,
- $\lambda_1 > \lambda^+$ *iff* $\lambda_2 > \lambda^+$,
- $\forall \lambda \in \left[\lambda^-, \lambda^+\right]$, $\lambda_1 = \lambda$ *iff* $\lambda_2 = \lambda$.

The existence of a Nash equilibrium of the supply game is guaranteed by the convexity of preferences and the (easily checked) fact that for any x, the boundary of each agent's option set is the graph of a strictly concave function: $\xi_1(\cdot, x_2)$ and $\xi_2(x_1, \cdot)$, respectively. Each agent has a unique best response to the other agent's strategy.

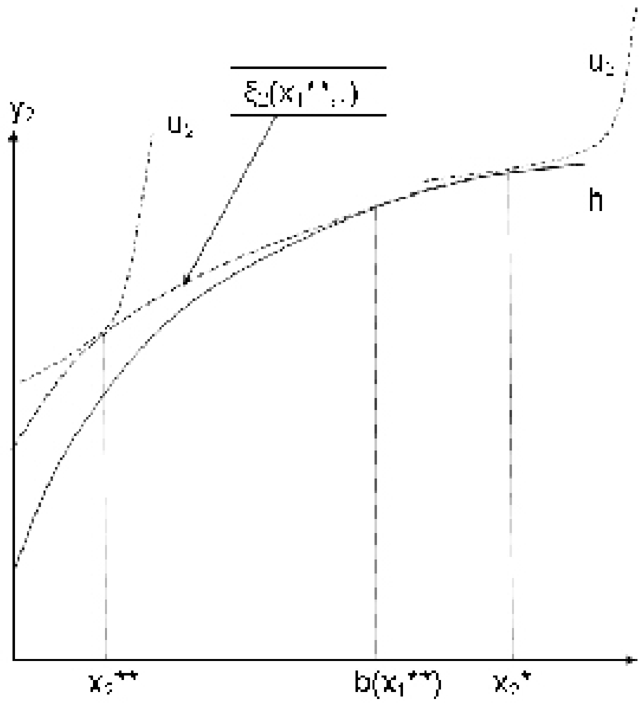

Figure 1: An impossible configuration.

We show uniqueness by contradiction. Fix a preference profile, and suppose the supply game admits two distinct Nash equilibria, $x^*$and x^{**}, at that profile. We claim that $[b(x_1^*) \leq x_2^* \implies b(x_1^{**}) > x_2^{**}]$ and $[b(x_1^*) \geq x_2^* \implies b(x_1^{**}) < x_2^{**}]$. Suppose not, and suppose without loss of generality that $b(x_1^*) \leq x_2^*$ and $b(x_1^{**}) \leq x_2^{**}$. Then, if $\xi \in \mathcal{R}$, observe that $\xi_2(x_1^*, \cdot) \equiv \xi_2(x_1^{**}, \cdot) \equiv h^r(\cdot)$ on the interval $[\max\{b(x_1^*), b(x_1^{**})\}, +\infty[$ (and therefore on any closed subinterval). By Lemma 1, $x_2^* = x_2^{**}$. Because agent 1 has a unique best response to x_2^* (and to x_2^{**}) it follows that $x_1^* = x_1^{**}$, contradicting the assumption that x^* and x^{**} are distinct. If $\xi \in \mathcal{S}$, the argument is similar upon noticing that $\xi_1(\cdot, x_2^*) \equiv \xi_1(\cdot, x_2^{**}) \equiv g^s(\cdot)$ on the interval $[0, \min\{x_2^*, x_2^{**}\}]$.

It follows from the argument of the previous paragraph that the supply game induced by ξ has a unique Nash equilibrium at all profiles if $\xi \in \mathcal{D}$. Next, we only show uniqueness for the case $\xi \in \mathcal{R}\backslash\mathcal{D}$ as the argument is similar for $\xi \in \mathcal{S}\backslash\mathcal{D}$.

Let $\xi \in \mathcal{R}\backslash\mathcal{D}$ and suppose without loss that $b(x_1^*) \leq x_2^*$ and $b(x_1^{**}) > x_2^{**}$. We claim that $b(x_1^{**}) \geq x_2^*$. If not, then $x_2^* > b(x_1^{**}) > x_2^{**}$ and $\xi_2(x_1^*, \cdot) \equiv \xi_2(x_1^{**}, \cdot) \equiv h^r(\cdot)$ on $[\max\{b(x_1^*), b(x_1^{**})\}, +\infty[$; see Figure 1 (drawn for the case $b(x_1^{**}) \geq b(x_1^*)$). It follows from Lemma 1 and $x_2^* \geq \max\{b(x_1^*), b(x_1^{**})\}$ that $x_2 \geq b(x_1^{**})$, a contradiction of our assumption. We can show similarly that $b(x_1^{**}) \leq x_2^*$ (and hence $b(x_1^{**}) = x_2^*$) and that $b(x_1^*) = x_2^{**}$. Finally, one can apply Lemma 1 one last time to contradict $b(x_1^{**}) = x_2^* > b(x_1^*) = x_2^{**}$.

iii) $\implies$ *ii)* Obvious.

ii) $\implies$ *i)* Let ξ be a sharing rule for which the associated supply game has at most one Nash equilibrium at all profiles, we show $\xi \in \mathcal{S} \cup \mathcal{R}$.

Notation: We say that a 2×2 matrix, $[\alpha_{ij}]$, is *acyclic* if $\alpha_{12}\alpha_{21} = 0$. We say that a sharing rule, ξ, is *acyclic at a point* $x \in \mathbb{R}_+^2$ if the Jacobian matrix of ξ at x, $\left[\frac{\partial \xi_i}{\partial x_j}(x)\right]$, is acyclic. We define $NE = \left\{x \in \mathbb{R}^2_+ | \frac{\partial \xi_1}{\partial x_2}(x) \neq 0 \text{ and } \frac{\partial \xi_2}{\partial x_1}(x) = 0\right\}$, $SW = \left\{x \in \mathbb{R}^2_+ | \frac{\partial \xi_1}{\partial x_2}(x) = 0 \text{ and } \frac{\partial \xi_2}{\partial x_1}(x) \neq 0\right\}$ and $D = \left\{x \in \mathbb{R}^2_+ | \frac{\partial \xi_1}{\partial x_2}(x) = \frac{\partial \xi_2}{\partial x_1}(x) = 0\right\}$.

We start the proof by restating a lemma from the proof of Theorem 2 in Moulin and Shenker (1992), which still holds in our setting. It is related to the finding that strategyproof mechanisms must be acyclic at differentiable points (see Satterthwaite and Sonnenschein, 1981).

Lemma 2 *(Lemma 5 in [12]) If the supply game associated with ξ has at most one Nash equilibrium, then ξ is acyclic at all $x \in \mathbb{R}^2_+$.*

It is clear from acyclicity that NE, SW and D form a partition of $\mathbb{R}^2_+$ and from smoothness that NE and SW are open whereas D is closed.

Claim 1 *a) ξ can be written as*

$$\xi(x) = (g(x_1), F(x_1 + x_2) - g(x_1))$$

on any connected open subset of SW for some mapping $g \in \mathcal{F}$.

b) ξ can be written as

$$\xi(x) = (F(x_1 + x_2) - h(x_2), h(x_2))$$

on any connected open subset of NE for some mapping $h \in \mathcal{F}$.

Proof. We only prove statement a). By definition, $\frac{\partial \xi_1}{\partial x_2} \equiv 0$ on SW, which implies that for any connected open subset, Σ, of SW there exists a mapping g such that $\xi_1(x) = g(x_1)$ on Σ; also, budget balance requires $\xi_2(x) = F(x_1 + x_2) - g(x_1)$ on Σ. The monotonicity and smoothness of ξ imply that g must be strictly increasing and continuously differentiable, respectively.

It remains to show the strict concavity of g. Consider any $x \in \Sigma$ and $\varepsilon > 0$ such that the closed ball, $\bar{B}(x, \varepsilon)$, is included in Σ and suppose g is convex on $[x_1 - \varepsilon, x_1 + \varepsilon]$. Because for any $s \in [x_1 - \varepsilon, x_1 + \varepsilon]$, $\xi_2(s, \cdot) \equiv F(s + \cdot) - g(s)$ is strictly concave, one can find a utility function u_2 such that x_2 is agent 2's best response to any $s \in [x_1 - \varepsilon, x_1 + \varepsilon]$.[4] Then, one can construct a utility function u_1 such that $u_1(x_1 - \varepsilon, g(x_1 - \varepsilon)) = u_1(x_1 + \varepsilon, g(x_1 + \varepsilon)) = \max_{[x_1 - \varepsilon, x_1 + \varepsilon]} \{u_1(s, g(s))\}$; see Figure 2. By monotonicity of $\xi_1(\cdot, x_2)$ and convexity of preferences, one can find u_1 "convex enough" such that $u_1(x_1 - \varepsilon, g(x_1 - \varepsilon)) = u_1(x_1 + \varepsilon, g(x_1 + \varepsilon)) = \max_{\{s | (s, x_2) \in \Sigma\}} \{u_1(s, \xi_1(s, x_2))\}$. Hence, both $(x_1 - \varepsilon, x_2)$ and $(x_1 + \varepsilon, x_2)$ are Nash equilibria of the supply game, contradicting the uniqueness assumption. ■

The remainder of the proof consists in establishing that the boundary between NE and SW is increasing and is the graph of some increasing real-valued function. But first we must make sure that this boundary exists.

Claim 2 *i) D has empty interior,*

ii) there exists a boundary, B, between NE and SW, if both are nonempty,

iii) $B \subset D$.

Proof. Suppose i) is false and consider an open neighborhood in D containing 4 points x^A, x^B, x^C, x^D such that x^C (resp. x^D) lies North of x^A (resp. x^B) and x^B (resp. x^D) lies East of x^A (resp. x^C); see Figure 3.

[4] E.g. by making the indifference curves of agent 2's preference arbitrarily close to being right-angled.

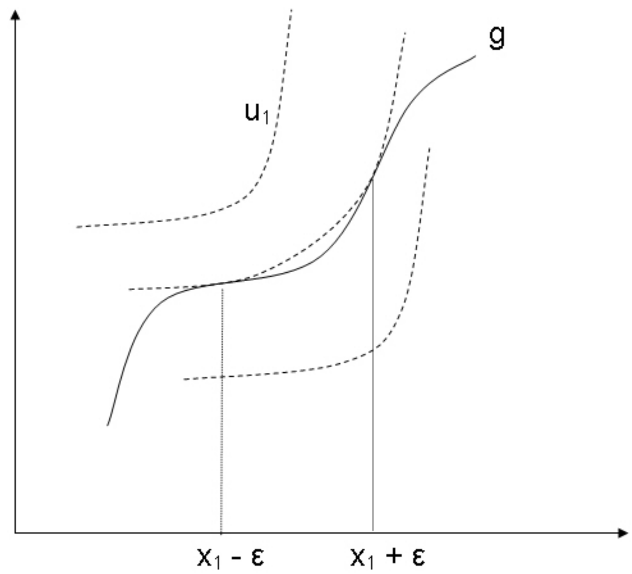

Figure 2: Multiple equilibria may exist if g is not strictly concave.

Notice that budget balance requires $\xi_1(x) + \xi_2(x) = F(x_1 + x_2)$ for all x, which implies:

$$\frac{\partial \xi_1}{\partial x_1}(x) + \frac{\partial \xi_2}{\partial x_1}(x) = F'(x_1 + x_2) \tag{1}$$

for all $x \in \mathbb{R}^2_+$. Therefore, $\frac{\partial \xi_1}{\partial x_1}(x) = F'(x_1 + x_2)$ on D. Thus, taking the integral between x^A and x^B yields $\xi_1(x^B) = \xi_1(x^A) + F(x_1^B + x_2^B) - F(x_1^A + x_2^A)$; also, $\xi_1(x^D) = \xi_1(x^B)$ because $\frac{\partial \xi_2}{\partial x_1} \equiv 0$ on D. Therefore $\xi_1(x^D) = \xi_1(x^A) + F(x_1^B + x_2^B) - F(x_1^A + x_2^A)$. Similarly, $\xi_1(x^D) = \xi_1(x^C) + F(x_1^D + x_2^D) - F(x_1^C + x_2^C) = \xi_1(x^A) + F(x_1^D + x_2^D) - F(x_1^C + x_2^C)$, which implies

$$F(x_1^B + x_2^B) - F(x_1^A + x_2^A) = F(x_1^D + x_2^D) - F(x_1^C + x_2^C),$$

a clear contradiction of the strict concavity of F. Therefore D is of empty interior and the boundary between NE and SW exists; smoothness implies that the latter is contained in D. ■

We call any continuous path of $\mathbb{R}^2_+$ which is a subset of B a *portion* of the boundary.

Claim 3 *B does not contain vertical or horizontal portions.*

Proof. Suppose B contains a non-degenerate horizontal portion $\left[x_1^-, x_1^+\right] \times \{x_2\}$. Smoothness, along with the fact that $B \subset D$ and Claim 1 imply $h'(x_2) = g'(x_1)$ for all x_1 in some non-degenerate sub-interval of $\left[x_1^-, x_1^+\right]$ for some mappings $g, h \in \mathcal{F}$, contradicting the strict concavity of g. Similarly, B cannot contain a vertical portion. ■

Claim 4 *On a portion of B, x_2 increases with x_1.*

Proof. As above, $h'(x_2) = g'(x_1)$ must hold at any point on the boundary and the claim holds true by strict concavity of g and h. ■

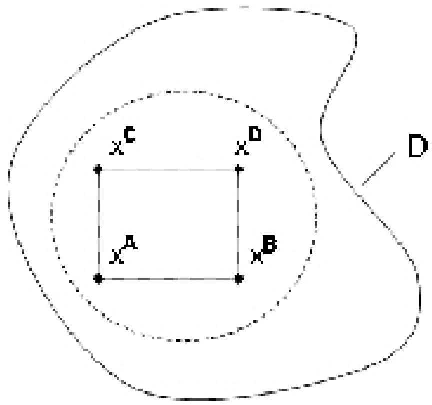

Figure 3:

We introduce some more notation. We define the sets $SW^+ = \left\{x \in SW | \frac{\partial \xi_2(x)}{\partial x_1} > 0\right\}$ and $SW^- = \left\{x \in SW | \frac{\partial \xi_2(x)}{\partial x_1} < 0\right\}$; the sets NE^+ and NE^- are similarly defined. By smoothness, these four sets are open.

Claim 5 *a) SW^- is north-comprehensive: $SW^- + \{0\} \times \mathbb{R}_+ \subset SW^-$,*

b) SW^+ is south-comprehensive: $SW^+ + \{0\} \times \mathbb{R}_- \subset SW^+$,

c) NE^- is east-comprehensive: $NE^- + \mathbb{R}_+ \times \{0\} \subset NE^-$,

d) NE^+ is west-comprehensive: $NE^+ + \mathbb{R}_- \times \{0\} \subset NE^+$.

Proof. We only prove statement a). Let $x \in SW^-$, and consider an open neighborhood of x contained in SW^-. On that neighborhood, $\frac{\partial \xi_1}{\partial x_2} = 0$, i.e. ξ_1 is independent of x_2; in particular, the ratio $\frac{\xi_1(x_1+\varepsilon, x_2) - \xi_1(x_1, x_2)}{\varepsilon}$ is also independent of x_2 on that neighborhood for small values of ε. Taking the limit, $\frac{\partial \xi_1}{\partial x_1}$ is independent of x_2 on a neighborhood of x. By the strict concavity of F, expression (1) implies that $\frac{\partial \xi_2}{\partial x_1}$ must be decreasing in x_2. In addition, because $x \in SW^-$, we have $\frac{\partial \xi_2}{\partial x_1}(x) < 0$; it follows that $\frac{\partial \xi_2}{\partial x_1}(x_1, x_2 + \lambda) < 0$ for any $\lambda > 0$. Thus, by acyclicity, SW^- is north-comprehensive. ■

More notation. We denote by SW^-/NE^- a portion of B with SW^- (resp. NE^-) in the immediate northwest (resp. southeast) vicinity of the boundary. NE^+/SW^+ portions are similarly defined.

Claim 6 *B consists only of SW^-/NE^- and NE^+/SW^+ portions.*

Proof. From the previous claim. Because SW^- is north-comprehensive and because B is included in D, there cannot be any points in SW^- south of B. Similarly, there cannot be any points in SW^+ (resp. NE^-, NE^+) north (resp. west, east) of B.

We complete the proof of the claim by showing that any portion of B containing a SW^-/NE^- (resp. NE^+/SW^+) subportion must be a SW^-/NE^- (resp. NE^+/SW^+) portion. Suppose there

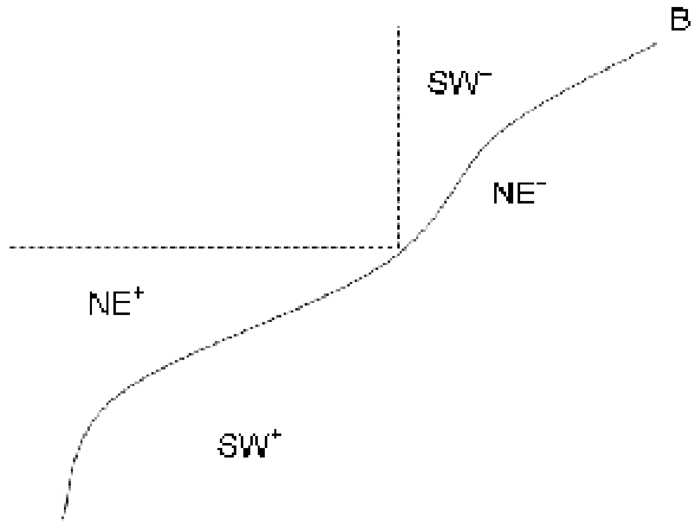

Figure 4:

exists a portion of B containing both a SW^-/NE^- and a NE^+/SW^+ subportion. By the comprehensiveness of SW^- and NE^+, it must be that the SW^-/NE^- subportion lies to the northeast of the NE^+/SW^+ subportion (see Figure 4). Yet, the north-comprehensiveness of SW^- and the west-comprehensiveness of NE^+ imply that there exists a horizontal, vertical or decreasing portion of B; a contradiction (Claim 4). ■

Claim 7 *No point of B can lie northwest of another.*

Proof. Suppose the claim is not true and let x, $x' \in B$ such that $x'_1 > x_1$ and $x'_2 < x_2$.[5] The reader can check that by Claim 5, x and x' belong to two portions of different type. W.l.o.g. assume x belongs to a NE^+/SW^+ portion and x' belongs to a SW^-/NE^- portion.

Define $\bar{x}_1 > x_1$ to be the smallest real number such that $(\bar{x}_1, t) \notin SW^+$ for any $t \geq 0$ if such a number exists; if not, define $\bar{x}_1 = x'_1$. Denote by $b : [x_1, \bar{x}_1[\to \mathbb{R}_+$ the function whose graph is the north boundary of SW^+. Note that by the south-comprehensiveness of SW^+, b is well defined on $[x_1, \bar{x}_1[$; and by the smoothness of ξ, b is continuous.

We show that b defines a NE^+/SW^+ portion on every interval where it is increasing. Indeed, suppose there exists $x_1^* \in]x_1, \bar{x}_1[$ and $\varepsilon > 0$ such that the immediate vicinity north of the graph of b, $\mathcal{V} \equiv B((x_1^*, b(x_1^*)), \varepsilon) \cap \{x | x_1 \in [x_1, \bar{x}_1[, x_2 > b(x_1)\}$, does not intersect NE^+; i.e., such that

$$\mathcal{V} \cap NE^+ = \emptyset.$$

Then claims 5 and 6 imply $\mathcal{V} \subset SW^-$, which leads to a contradiction by implying the existence of a horizontal, vertical or decreasing portion of B as in the proof of the previous claim.

We now show that b must be increasing on its domain. Indeed, the immediate vicinity north of the graph of b cannot intersect NE^+ on a non-degenerate non-increasing interval of b (Claim 4), nor can it intersect NE^- (by east-comprehensiveness of NE^-, north-comprehensiveness of SW^- and

[5] Clearly, Claim 5 implies $x'_1 \neq x_1$ and $x'_2 \neq x_2$.

the fact that x' belongs to a SW^-/NE^- portion) or SW^+ (by definition of b); hence it is a subset of SW^-, thus leading to the same contradiction as in the previous paragraph.

Finally, because b is increasing on $[x_1, \bar{x}_1[$, $\bar{x}_1$ is indeed the smallest number such that $(\bar{x}_1, t) \notin SW^+$ for any $t \geq 0$ (the north-comprehensiveness of SW^- implies this fact even if $\bar{x}_1$ was originally taken to be equal to x_1'). It follows that a subset of $\{\bar{x}_1\} \times \mathbb{R}_+$ belongs to the boundary of SW^+, contradicting the fact that SW^+ cannot have a vertical boundary (easily proved, as in Claim 3). ■

Claim 8 *B is the graph of a non-decreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ which is continuous and increasing on $\{t \geq 0 | 0 < b(t) < +\infty\}$.*

Proof. If $NE = \emptyset$ or $SW = \emptyset$, the boundary B is vacuously defined: set $b \equiv 0$ or $b \equiv +\infty$.

If $NE \neq \emptyset$ and $SW \neq \emptyset$, define the set $X_1 = \{x_1 \in \mathbb{R}_+ | \exists x_2 > 0 \text{ s.t. } (x_1, x_2) \in B\}$. By the previous claims (2, 4 and 7), X_1 is an interval and there exists a continuous and increasing function $b : X_1 \to \mathbb{R}_+$ whose graph is $B \cap (X_1 \times \mathbb{R}_+)$. We extend the domain of b by setting $b(x_1) = 0$ for all $x_1 \leq \inf X_1$ if $\inf X_1 > 0$ and by defining $b(x_1) = +\infty$ for all $x_1 \geq \sup X_1$ if $\sup X_1$ exists. ■

Conclusion of the proof: The statement of Theorem 1 follows from claims 1 and 8.

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