Pooling private technologies: Improving upon autarky*

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Abstract

When n agents decide to pool their private, decreasing returns technologies, *single-path methods* are a natural way to share joint output because of their strong incentives properties (Friedman, 2002). They are a non-anonymous generalization of the serial rule (Moulin and Shenker, 1992) sharing a production function along a prespecified path. We show that only one of these methods satisfies voluntary participation; its generating path is entirely determined by the n production functions. This yields a bijection between single-path methods and distributions of property rights on a single technology. Also, we show that these methods are characterized by their incentives properties in the 2-agent case, but not for $n \geq 3$.

Keywords: Autarky, incentive compatibility, joint production, serial rule, technology pooling.

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1 Introduction

Several producers of a common private good decide to pool their private production possibilities. In addition to technological contributions, each producer makes input contributions to the cooperative. We assume that input is transferable across technologies. Two classical questions are: how to jointly utilize their private technologies and how to share the proceeds of their cooperation (see Israelsen [8], Sen [14], Weitzman [22])?

In a stylized version of the problem, each agent makes her privately owned machine (her technology) available to all the group-members and can supply labor (the input) to any machine. Practical examples include farmers pooling their land in a cooperative; here, land is the technology and input can be labor or seeds to be planted. Examples of such cooperatives can also be found in the fishing sector (Townsend [21]) and in the plywood industry (Craig and Pencavel [3]). A similar situation arises whenever a group of experts (e.g. lawyers, physicians, financial advisors, car salesmen, etc.) who can rank their clients in decreasing order of productivity decide to engage in a partnership; each agent's clientele then amounts to a decreasing-returns technology. By pooling their clienteles, the agents can reallocate their time or resources (the input) across the total pool of clients.

The first requirement is that production possibilities and input contributions be pooled efficiently. When returns to scale vary, there is typically a unique efficient way to reallocate a given amount of input across the various technologies.[1] Thus, the autarkic use of the production possibilities, where agent i only supplies input to her own technology, can be Pareto-improved. The *aggregate* production function (of the individual technologies) summarizes these production opportunities.

We assume that the individual technologies are known to the planner and exhibit decreasing returns to scale. Information about the preferences of the agents is private, potentially leaving room for misrepresentation. We design a method to share the total output between the n agents according to their (technological and input) contributions while insisting on two requirements: one strategic, the other normative. First, we require that the agents' incentives to contribute input be unambiguous; by *incentive compatibility* we mean Nash-implementability with unique equilibrium[2]. We also demand voluntary participation, i.e., every

[1] For instance, if machine 1 is always more productive than the others, productive efficiency requires that the $(n-1)$ other agents work on machine 1 instead of their own.

[2] In this model the corresponding direct revelation mechanism is then group-strategyproof;

agent should be at least as well off under the pooling method as by reverting to her own technology; we refer to this condition as *autarkic individual rationality* (a term introduced by Saijo [12] in the public good context).

The contribution of this paper is twofold. We first examine the situation where a single production function is to be shared when no property rights are assigned. The serial rule (Moulin and Shenker, [11]) treats all agents equally and hence is one interpretation of equal rights on the technology. Friedman ([5]) gives a non-anonymous generalization of the serial rule which shares the production function, F, along a predetermined path in the input space of the agents, ϕ. The corresponding "single-path method" works as follows[3]. Each agent i chooses her level of input, x_i. The resulting total output level, $F\left(\sum_i x_i\right)$, is shared incrementally. Output from the first unit of labor supply, $F(1)$, is shared in proportion to $(\phi_1(1), \phi_2(1), ..., \phi_n(1))$, which can be interpreted as a "labor responsibility" vector; agent i is "responsible" for the fraction $\phi_i(1)$ of the first unit of society's labor, and is rewarded accordingly.

Output from the second unit of labor, $F(2) - F(1)$, is then shared in proportion to each $\phi_i(2) - \phi_i(1)$, the increment of labor "demanded" from each agent i. We continue allocating output according to the direction of the path, ϕ', until $\phi_i(t) = x_i$ for some i; i.e. until the amount of labor "demanded" from agent i equals the amount she chose to supply. Agent i then leaves the procedure with her reward and the remaining agents continue to share output in proportion to the ϕ_j's. And so on.

Clearly, agent i's reward depends not only on x_i but also on the labor contributions of other agents. We illustrate the induced game on a 3-person example, from the point of view of player 1. Suppose ϕ is the straight line from the origin with direction $\overrightarrow{w} = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$, such that $\phi(t) = t\overrightarrow{w}$. Because $w_1 = \frac{1}{6}$, agent 1 is entitled to "one sixth of F" in the sense that she is guaranteed $\frac{1}{6}F(6x_1)$ units of output. She will receive exactly that amount unless another agent, say agent 3, leaves the procedure first (which happens if and only if $x_3 < 3x_1$ and $2x_3 < 3x_2$). At the point where agent 3 is served, agent 1 has already provided $\frac{x_3}{3}$ units of labor and agent 2 has supplied $\frac{2x_3}{3}$ units $\left(= \frac{w_2}{w_3}x_3\right)$; agent 1 has received $\frac{1}{6}F(2x_3)$, one sixth of total output.

The remainder of the technology, $\bar{F} = F(2x_3 + \cdot) - F(2x_3)$, is then shared between the remaining agents 1 and 2 in proportion to (w_1, w_2): agent 1 is entitled to "one third of $\bar{F}$" $\left(\frac{w_1}{w_1+w_2} = \frac{1}{3}\right)$ in addition to the output she has

see below for a discussion of other, weaker interpretations of incentive compatibility.

[3]Friedman uses the term "fixed-path methods".

already secured: $\frac{1}{6}F(2x_3)+\frac{1}{3}\bar{F}\left(3\left(x_1-\frac{x_3}{3}\right)\right)$. The concavity of F ensures that this reward is better than getting "one sixth of F" throughout. Finally, if agent 2 exits the procedure before agent 1 (i.e. if $x_2<2x_1$), agent 1 receives the full remainder of F from that point on, $F(x_1+x_2+x_3)-F\left(x_3+\frac{3}{2}x_2\right)$, in addition to the output she has already secured.

Our first result is a direct generalization of Moulin and Shenker's characterization of the serial rule in the two-person case (Theorem 2). They show that the serial rule is the only anonymous and incentive compatible sharing rule. By contrast, we drop anonymity and instead merely require that an agent contributing no input be not rewarded (our *zero output for zero input* condition, *ZOZI*). While their result holds in the general case, ours does not extend to more than two agents (Theorem 2).

Our second main result establishes a close relationship between the problem of sharing a single technology and that of technology pooling for any number of agents. Technology pooling under autarkic individual rationality and incentive compatibility leads to a unique single-path method (Theorem 3). Its path is completely determined by the profile of production functions and the requirement to guarantee each agent her stand-alone utility level. The intuition is fairly simple and easily illustrated on the example above. Suppose three agents decide to pool production functions such that $f_1(t)=\frac{1}{2}f_2(2t)=\frac{1}{3}f_3(3t)$. Then, the labor responsibilities must be proportional to (1,2,3) at all levels t because $f_1'(t)=f_2'(2t)=f_3'(3t)$. Hence, the resulting path is the line borne by these weights, and F is the function determined by $F'(6t)=f_1'(t)=f_2'(2t)=f_3'(3t)$.

The paper is organized as follows. The next section relates our work to the existing literature. Section 3 sets up the single technology model and defines the class of single-path methods. Section 4 discusses the incentives properties of these methods. Section 5 justifies the use of single-path methods as compelling solutions to the issue of technology pooling and recommends a unique solution. In Section 6 we make the comment that if technologies were private information, this solution would be vulnerable to misrepresentations of technologies. Section 7 concludes. Most proofs can be found in the Appendix.

2 Relation to the literature

This work contributes to the large literature exploring the trade-off between efficiency and incentive compatibility in the production and distribution of private goods.

Our second result (Theorem 3) can be viewed as a follow-up on work by Friedman ([5],[6]) in the sense that we provide a motivation for single-path methods that was lacking there.

Next, a series of characterization results (de Frutos, [7], Moulin and Shenker, [11], and Moulin, [10]) is closely related to our Theorem 2. The first statement of Theorem 2 is a non-anonymous generalization of Moulin and Shenker's characterization of the serial rule[4]. But our second statement suggests that many rules outside of the class of single-path methods meet our high standards of incentive compatibility for $n \geq 3$; their more complex path structure is similar to the "path functions" of Sprumont ([20]). This contrasts with Moulin ([10]) who shows in the discrete framework that single-path methods are in fact characterized by a incentive compatibility requirement fairly close to ours. This discrepancy illustrates a subtle difference between the discrete and continuous versions of the model and is worthy of future research.

Also, an interesting corollary of Theorems 2 and 3 is the existence of a unique incentive compatible (in the strongest sense) and (autarkically) individually rational pooling method in the 2-agent case (Corollary 1). This result is similar in spirit to Barberà and Jackson's characterization of strategyproof and individually rational allocation rules in exchange economies (in [1]). Like our solution, their "fixed-proportion trading" rules are not first-best efficient[5]. They also have a serial flavor as they are essentially an adaptation of Sprumont's ([19]) uniform rationing rule, where the amount to be rationed depends on the preference profile. This seriality is an oriented one, depending on whether there is excess supply or excess demand.

Recent related literature on the common production of private goods considers weaker interpretations of incentive compatibility (see, e.g., Corchón and Puy [2], Shin and Suh [17]). For instance, Corchón and Puy establish that any continuous sharing rule admits a Pareto-efficient allocation which can be Nash-implemented. Yet, any game implementing such an outcome must have several, non-welfare-equivalent Nash equilibria at some profiles. Here we insist on the uniqueness of the Nash equilibrium, a much more demanding requirement than regular Nash-implementability.

[4] The question remains open whether de Frutos' result can be generalized in the same way.

[5] Strategyproofness and efficiency are incompatible with individual rationality in exchange and production economies (see, respectively, Serizawa [15] and Leroux [9])

3 The single technology model

Let $N = \{1, ..., n\}$ be the set of agents. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a production function which is strictly increasing, strictly concave such that $F(0) = 0$. We denote by $\mathcal{F}$ the class of such functions. If in addition F is continuously differentiable, we write $F \in \mathcal{F}^c$. Each agent i provides a non-negative amount x_i of input to the common technology, and receive a non-negative quantity y_i of output such that $\sum_i y_i = F(\sum_i x_i)$. We write $x = (x_1, ..., x_n)$ and for any $i \in N$, (x'_i, x_{-i}) is the vector of inputs where the ith entry of x has been replaced by $x'_i \in \mathbb{R}_+$. A *bundle* is an element $z_i = (x_i, y_i) \in \mathbb{R}_+ \times \mathbb{R}$; we define an *allocation*, z, to be a list of n bundles, one for each agent. We denote by $Z_F = \left\{ z \in (\mathbb{R}_+ \times \mathbb{R})^N \mid \sum_i y_i \leq F(\sum_i x_i) \right\}$ the set of feasible allocations under F.

Each agent i can supply up to M_i units of input (with M_i possibly very large). Her *preferences* over bundles are defined on $\mathbb{R}_+ \times \mathbb{R}$; they are continuous, convex, strictly increasing in y_i, strictly decreasing in x_i and representable by a utility function u_i. While all our results are purely ordinal, we will use utility representations rather than the more cumbersome binary relation notation. We adopt the convention $u_i(x_i, y_i) = -\infty$ if $x_i > M_i$. We denote by $\mathcal{U}$ the class of preferences. A *preference profile* (or *utility profile*) is a list of n preferences, $u = (u_1, ..., u_n) \in \mathcal{U}^N$, one per agent. For any $j \in N$, we will sometimes abuse notations and write $u = (u_j, u_{-j})$.

Definition 1 *An F-*sharing method *(or F-*sharing rule*) is a mapping*

$$\begin{aligned} \xi : \quad & \mathbb{R}_+^N \to \mathbb{R}_+^N \\ & x \mapsto (\xi_1(x), ..., \xi_2(x)) \quad s.t. \quad \textstyle\sum_{i \in N} \xi_i(x) = F\left(\sum_{i \in N} x_i\right) \end{aligned}$$

that satisfies the follwing two properties:
—> Monotonicity: $\frac{\partial \xi_i}{\partial x_i} > 0$,
—> Zero output for zero input (ZOZI): $\forall x_{-i} \in \mathbb{R}_+^{N \setminus \{i\}} \quad \xi_i(0, x_{-i}) = 0.$
We denote by $\mathcal{S}_F$ the class of F-sharing rules.

If in addition $F \in \mathcal{F}^c$ and ξ satisfies the following smoothness property, we write $\xi \in \mathcal{S}_F^c$:
—> *Smoothness*: ξ is continuously differentiable on $\mathbb{R}_+^n$.

A few comments are in order. Monotonicity is a normatively appealing requirement. It states that an agent should receive strictly more output as her input contribution increases: it gives agents an incentive to supply input. Also, from the point of view of fairness, it implies that every agent will receive a positive fraction of the output resulting from her input contribution.

The normative aspect of the ZOZI property is twofold. The more obvious one is that an agent must provide some input in order to be rewarded. The other consequence is that an agent can always guarantee her utility level to be no less than $u_i(0,0)$ by choosing to supply nothing to the system ($x_i = 0$).

Finally, we demand that sharing methods be smooth. This requirement is a technical one. One of our proofs (Theorem 2) relies heavily on this assumption and, while we were not able to prove our results without imposing smoothness, we do not know whether it is a necessary condition. The same remark applies to results in Moulin and Shenker [11] and in Shenker [16].

For any preference profile $u \in \mathcal{U}^N$ and any F-sharing method $\xi \in \mathcal{S}_F$, we denote by $G(\xi; u)$ the *game* in which each agent's strategy space is $\mathbb{R}_+$ and agent i's payoff is $u_i(x_i, \xi_i(x))$ when x_j is the strategy played by agent $j \in N$.

We now define what we mean by "sharing a technology along a path". A *path* is a mapping

$$\begin{aligned} \phi : \quad & \mathbb{R}_+ \to \mathbb{R}_+^N \\ & t \mapsto (\phi_1(t), ..., \phi_n(t)) \end{aligned}$$

such that for all $i \in N$ the following two properties hold:

(a) ϕ_i is non-decreasing and differentiable on $\mathbb{R}_+$,

(b) $\sum_j \phi_j(t) = t$ for any $t \in \left[0, \sum_j M_j\right]$ and $\phi_i(t) = M_i$ for any $t \geq \sum_j M_j$.

We denote by $\mathcal{P}$ the class of paths. If a path ϕ also satisfies the following condition (c) for every i, we write $\phi \in \mathcal{P}^c$.

(c) $\phi_i'(t) = 0$ only if $\phi_i(t) = 0$ or $\phi_i(t) = M_i$.

Fix $\phi \in \mathcal{P}$. For any $i \in N$, define the mapping δ_i as follows:

$$\begin{aligned} \delta_i : \quad & [0, M_i] \to \mathbb{R}_+ \\ & x_i \mapsto \min\{t | \phi_i(t) \geq x_i\}. \end{aligned} \tag{1}$$

Because δ_i jumps wherever ϕ_i is flat on a non-degenerate interval, it is continuous on $]0, M_i]$ if and only if $\phi \in \mathcal{P}^c$.

Given a path $\phi \in \mathcal{P}$, we define the *single-path method generated by* ϕ, denoted ξ^ϕ, as follows. Let $x \in \times_i [0, M_i]$, without loss we relabel the agents such that $\delta_1(x_1) \leq \delta_2(x_2) \leq ... \leq \delta_n(x_n)$; i.e. such that the coordinates of x

are met along ϕ in the natural order. Let $t \geq 0$ be such that $\phi(t) \leq x$, i.e. such that no agent's supply level has yet been met. ξ^{ϕ} recommends that the marginal product $F'(t)$ be split between the agents according to the vector of proportions $\left(\phi_1'(t), ..., \phi_n'(t)\right)$ (recall that $\sum_i \phi_i'(t) = 1$). Once the input supply of the first agent is met along the path ($\phi_i(t) \geq x_i$), we freeze her output share and continue the sharing procedure with the remaining "active" agents. The subprocedure shares the remainder of F along the projection of ϕ on the subspace $\left\{s \in \mathbb{R}_+^n | s_1 = x_1\right\}$ until agent 2's supply is met. And so on. We next give a formal definition.

Definition 2 *The* single-path method generated by ϕ, *denoted* ξ^{ϕ}, *is the* F-*sharing rule defined by:*

$$\begin{array}{l} \xi_1^{\phi}(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_1(t) \\ \xi_2^{\phi}(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_2(t) + \int_{\delta_1(x_1)}^{\delta_2(x_2)} F'\left(x_1 + \sum_{i \geq 2} \phi_i(t)\right) d\phi_2(t) \\ \vdots \\ \xi_n^{\phi}(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_n(t) + ... + \int_{\delta_{n-1}(x_{n-1})}^{\delta_n(x_n)} F'\left(\sum_{i=1}^{n-1} x_i + \phi_n(t)\right) d\phi_n(t) \end{array} \tag{2}$$

for any $x \in \times_i [0, M_i]$.

A more compact notation is used by Friedman ([5], [6]): for any $i \in N$,

$$\xi_i^{\phi}(x) = \int_0^{\infty} F'\left(|\phi(t) \wedge x|\right) d(\phi_i(t) \wedge x_i)$$

where $|\cdot|$ returns the sum of the coordinates of a vector and $\wedge$ is the componentwise minimum of two vectors.

It follows easily from the monotonicity of F and the δ_j's that ξ^{ϕ} is monotonic ($\frac{\partial \xi_i^{\phi}}{\partial x_i} > 0$ for all i). Moreover, one can check (or see Friedman [5], Lemma 1) that ξ_i^{ϕ} is strictly concave in x_i. Because each function δ_j takes on the value zero at zero, the sharing rule ξ^{ϕ} satisfies the ZOZI condition. Hence, $\xi^{\phi} \in \mathcal{S}_F$. Finally, by inspecting formula (2) at the points x such that $\delta_i(x_i) = \delta_j(x_j)$ and those where $x_i = 0$, one can check that $\xi^{\phi} \in \mathcal{S}_F^c$ if $F \in \mathcal{F}^c$ and $\phi \in \mathcal{P}^c$. However, if $\phi \notin \mathcal{P} \backslash \mathcal{P}^c$, ξ^{ϕ} is not smooth.

When no confusion is possible, we will use the term "*ϕ-rule*" instead of the longer "single-path method". We next illustrate the definition of ϕ-rules with two examples:

Example 1: Incremental sharing. ($n = 2$) This method gives agent 1 full access to F; once agent 1 is served, agent 2 can use $F(x_1 + \cdot)$ at will. The corresponding path is

$$\phi^I : t \mapsto \begin{cases} (t, 0) & \text{if } t \leq M_1 < +\infty \\ (M_1, t - M_1) & \text{if } M_1 \leq t \leq M_1 + M_2 \end{cases}$$

i.e., ϕ^I is a parametrization of the horizontal axis up to $x_1 = M_1$. Output is awarded as follows:

$$\begin{cases} \xi_1^{\phi^I}(x) = F(x_1) \\ \xi_2^{\phi^I}(x) = F(x_1 + x_2) - F(x_1) \end{cases}$$

Example 2: Weighted serial rule. Assume $M_1 = M_2 = +\infty$[6]. Let $\alpha_1, ..., \alpha_n > 0$ and consider the path $\phi^S : t \mapsto (\alpha_1 t, ..., \alpha_n t)$. Let $x \in \mathbb{R}_+^N$ and assume without loss that $\frac{x_1}{\alpha_1} \leq \frac{x_2}{\alpha_2} \leq ... \leq \frac{x_n}{\alpha_n}$. Expression (2) then yields:

$$\xi_i^{\phi^S}(x) = \frac{\alpha_i}{\alpha^i} F(x^i) - \sum_{k=1}^{i-1} \frac{\alpha_i \alpha_k}{\alpha^k \alpha^{k+1}} F(x^k) \quad \text{for all } i = 1, ..., n,$$

where $\alpha^k = \sum_{j=k}^n \alpha_j$, and $x^k = x_1 + ... + x_{k-1} + \frac{\alpha^k}{\alpha_k} x_k$. As a particular case, the usual serial rule assigns identical weight to each agent.

4 Incentives properties of single-path methods

The family of ϕ-rules was introduced in Friedman [5] as a non-anonymous generalization of the serial rule retaining the latter's strong incentive properties.

Theorem 1 *Let ξ be a ϕ-rule, the following statements are true:*
i) $G(\xi; u)$ has a unique Nash equilibrium,
ii) every Nash equilibrium of $G(\xi; u)$ is strong.

Proof. It is shown in Friedman [5] that for any production function $F \in \mathcal{F}$, any path $\phi \in \mathcal{P}$ and any preference profile $u \in \mathcal{U}^N$, the game induced by ξ^ϕ satisfies a more demanding equilibrium property called O-solvability. ■

It follows from a standard result of the implementation literature (see Theorem 7.2.3 in Dasgupta et al. [4]) that the associated direct revelation mechanism is group-strategyproof.

[6] Although M_1 and M_2 were defined as real numbers, the definition of the weighted serial rule readily extends to the case where they are infinite.

Moulin and Shenker ([11]) established that the serial rule could be characterized by the equilibrium properties of Theorem 1 along with Anonymity ($x_i = x_j \implies \xi_i(x) = \xi_j(x)$). Concerning the natural question of whether the former properties alone characterize the class of smooth single-path methods, the result turns out to be true for the 2-agent case, but not for $n > 2$.

Theorem 2 *I. Assume $n = 2$ and $F \in \mathcal{F}^c$. The following statements are equivalent for any $\xi \in \mathcal{S}_F^c$:*
i) $G(\xi; u)$ has a unique Nash equilibrium,
ii) every Nash equilibrium of $G(\xi; u)$ is strong,
iii) ξ is a single-path method: $\exists \phi \in \mathcal{P}^c$ s.t. $\xi \equiv \xi^\phi$.
II. Statement I does not hold if $n > 2$.

Proof. The proof of statement I can be found in Appendix A.1. The methodology of the proof is related to that of Theorem 2 in Moulin and Shenker [11] and makes use of the acyclicity of strategyproof sharing rules (see Satterthwaite and Sonnenschein [13])

Proof of II. Assume $n = 3$, $F \in \mathcal{F}^c$ and let $\phi \in \mathcal{P}^c$. Consider an F-sharing rule ξ that coincides with ξ^ϕ until one of the agents is served, say agent i, but then shares the remainder of F between the remaining two agents along a strictly increasing subpath, $\psi(i, x_i)$, depending on the identity of the first-served agent and her input supply level. Note that $\psi(i, x_i)$ may differ from the projection of ϕ onto the plane $\left\{s \in \mathbb{R}_+^n | s_i = x_i\right\}$ for some pair (i, x_i). It is clear that agent i has the same unique dominant strategy under ξ and under ξ^ϕ. A straightforward application of Theorem 1 yields that the remaining agents also have a unique dominant strategy regardless of ψ. Hence, ξ satisfies the provisos i) and ii) of Theorem 2; also, $\xi \in \mathcal{S}_F^c$ (left to the reader). Yet, ξ is *not* a single-path method. Note that when $n = 2$, the type of methods just described cannot be distinguished from single-path methods. ■

Remark 1 *In the discrete version of our model, Moulin ([10]) establishes that "incremental sharing rules" (the discrete equivalent of single-path methods) are characterized by similar strategic properties for any number of agents. Interestingly, the continuous framework allows for a much richer class of incentive compatible rules.*

We show on a straighforward example why some of these more complex rules do not meet our incentive compatibility requirement in the discrete setting.

Consider a technology given by the discrete increments $\partial F : 4, 2, 1, 0$ (i.e. $F(1) = 4$, $F(2) = 4 + 2$,...) to be shared between 3 agents, each of whom can supply 0 or 1 unit of input. Suppose that the path structure used to share F yields the following priority orderings: $1\to 2 \to 3$ if $x_1 = 1$ and $1\to 3 \to 2$ if $x_1 = 0$. If preferences are such that agent 1 is indifferent between bundles (1,4) and (0,0), and if agent 2 prefers (1,2) to (0,0), then agent 1 can help out agent 3 by deciding not to work, thus giving her access to the bundle (1,2) instead of (1,1).

The above rule is immune to coalitional deviations in a weak sense: at least one agent in the deviating coalition does not strictly benefit (agent 1). Yet, not every Nash equilibrium of the supply game is strong due to agent 1's indifference between two bundles. Such indifferences are ruled out by the specifications of the continuous model.

The path structures described in the proof of statement II are what Sprumont calls "path functions" in [20], though his use of these path functions is ultimately quite different from ours. A natural question is to ask whether sharing rules generated by these path structures exhaust the class of incentive compatible methods (in the sense of provisos i) and ii)).

5 Pooling private technologies

Consider a situation where each agent privately owns a technology, $f_i \in \mathcal{F}$, which she decides to contribute to a cooperative along with an amount of input $x_i \in [0, M_i]$. One can think of the individual technologies as being machines and input being labor time. Labor is transferable, meaning that agents are able to work on machines other than their own. The manager of the cooperative (the planner) allocates the labor time of the workers across the various machines; e.g., if $x_1 = 3$, agent 1 may be asked to spend, say, two units of input on machine 1 and one unit on machine 4. The resulting total output is distributed between the agents according to their labor (the x_i's) and technological (the f_i's) contributions. Technologies are known to the planner, but the preferences of the agents are private information.

Define F^* to be the *aggregated production function* resulting from the efficient usage of the combined individual technologies:

$$\forall t \in \mathbb{R}_+ \quad F^*(t) = \max_{\substack{(x_1, ..., x_n) \in \mathbb{R}_+^N \\ \sum_i x_i = t}} \sum_{i=1}^{n} f_i(x_i). \tag{3}$$

Notice that because the f_i's belong to $\mathcal{F}$, F^* must also belong to $\mathcal{F}$. Also, if all the f_i's belong to $\mathcal{F}^c$, so does F^* (the reader can check that the converse is not true). We call $f = (f_1, ..., f_n) \in \mathcal{F}^N$ the *technology profile*.

Thus, the pooling framework is tantamount to the previous context of sharing a single technology. Here, however, autarkic individual rationality is a concern: no agent should be better off by using her private technology on her own. This voluntary participation requirement will end up determining uniquely the single-path method to use.

Definition 3 *An f-*pooling method *is an F^*-sharing rule ξ such that for any preference profile u and any Nash equilibrium x^* of $G(\xi; u)$ the following holds:*

$$u_i\left(x_i^*, \xi_i(x^*)\right) \geq sa_i(u_i) \equiv \max\left\{u_i(x_i, y_i) | y_i \leq f_i(x_i)\right\} \qquad \forall i \in N. \tag{4}$$

We say that ξ pools *f and we denote by $\mathcal{S}_f$ the class of f-pooling methods. If moreover ξ is smooth, i.e. if $\xi \in S^c_{F^*}$, we write $\xi \in S^c_f$.*

Define by ϕ^* the mapping assigning to each $t \geq 0$ the unique solution vector of (3); notice that ϕ^* is a path. The following theorem motivates the use of single-path methods.

Theorem 3 *ξ^{ϕ^*} is the unique single-path method which pools f.*

The following important corollary follows immediately from Theorems 2 and 3.

Corollary 1 *Assume $n = 2$ and $f \in (\mathcal{F}^c)^N$. ξ^{ϕ^*} is the unique smooth and incentive compatible (in the sense of provisos i) and ii)) f-pooling method.*

The following comments concerning ϕ^* will prove useful. Because ϕ^* is the unique solution of expression (3), it follows that

$$F^{*\prime}(t) = f_i'(\phi_i^*(t)) \tag{5}$$

whenever $\phi_i^*(t) > 0$ (technology i is in use). I.e., $\phi_i^*(t)$ is the level of input that can be used on technology i before its productivity falls below $F^{*\prime}(t)$. Hence, for a given $t > 0$, the larger $\phi_i^*(t)$, the more productive technology i is.

We now give some intuition as to why ξ^{ϕ^*} not only satisfies AIR but also improves upon autarky. As long as all agents are active ($t \leq \min_j \delta_j^*(x_j)$), ξ^{ϕ^*} shares the marginal product $F^{*\prime}(t)$ according to the vector of ratios $\left(\phi_1^{*\prime}(t), ..., \phi_n^{*\prime}(t)\right)$. Hence, assuming for clarity that $\delta_1^*(x_1)$ is the smallest of the $\delta_j^*(x_j)$'s, then

$$\xi_1^{\phi^*}(x) = \int_0^{\delta_1^*(x_1)} F^{*\prime}(t)\phi^{*\prime}(t)dt = f_1(x_1)$$

and agent 1 receives her stand-alone level of output. Now, for $\delta_1^*(x_1) \leq t \leq \min_{j \neq 1} \delta_j^*(x_j)$, ξ^{ϕ^*} shares the marginal output $F^{*\prime}(t)$ between agents 2,...,n according to the ratios $\left(\phi_2^{*\prime}(t), ..., \phi_n^{*\prime}(t)\right) \times \frac{1}{\sum_{j>1} \phi_j^{*\prime}(t)}$. Clearly, for any $i \neq 1$, $\frac{\phi_i^{*\prime}(t)}{\sum_{j>1} \phi_j^{*\prime}(t)} \geq \phi_i^{*\prime}(t)$ and agent i receives no less (typically more) than her stand-alone share of output. And so on. Improvement upon autarky obtains by integration. In words, when an agent leaves the procedure what is left of her technology is shared between the remaining agents in proportion to their technological contributions to F^*. The formal proof of Theorem 3 can be found in Appendix A.2.

Remark 2 *Among the rules generated by path stuctures as in Sprumont [20], all those (and only those) whose main path is ϕ^* are f-pooling methods, but their subpaths may be arbitrary. Thus, these rules may lack an internal consistency of sorts, unlike ξ^{ϕ^*}.*

Theorem 3 has an interesting converse interpretation. Given a production function F^*, to any path ϕ^* corresponds a unique decomposition of F^* into a "virtual" production profile, f, such that ξ^{ϕ^*} is the unique single-path method pooling f.

Theorem 4 *For any $F^* \in \mathcal{F}$ and any $\phi^* \in \mathcal{P}$, there exists a unique technology profile f decomposing F^* in the sense of (3) such that ξ^{ϕ^*} pools f. For any $i \in N$, f_i is given by*

$$f_i(x_i) = \int_0^{x_i} F^{*\prime}(\delta_i^*(t))dt$$

for all $0 \leq x_i \leq M_i$, where δ_i^* is defined relative to ϕ_i^* as in expression (1).

Theorems 3 and 4 together establish a striking bijection between the family of ϕ-rules and the possible distribution of property rights on F^*.

Proof. Immediate from Theorem 3. Let $F^* \in \mathcal{F}$, $\phi^* \in \mathcal{P}$ and $f \in \mathcal{F}^N$ decomposing F^* in the sense of (3) such that ξ^{ϕ^*} pools f. For any $i \in N$, expression (5) holds almost everywhere. I.e.,

$$f_i'(t) = F^{*\prime}(\delta_i^*(t)) \quad \text{almost everywhere.}$$

The result follows from integrating between 0 and x_i (recall $f_i(0) = 0$). ■

To illustrate Theorem 4, we provide the virtual production profiles corresponding to examples of Section 3.

Example 1. ξ^{ϕ^I} gives priority to agent 1. It is equivalent to pooling the production profile where agent 2's technology is useless compared to that of agent 1 on $[0, M_1 + M_2]$.

Example 2. Agents contribute to F^* in proportion to the α_i's: $f_i(t) = \alpha_i F^*(\frac{t}{\alpha_i})$.

6 Manipulation via misrepresentation of technology

Throughout the paper we assumed the private technologies to be known to the planner while the possibility of strategic manipulation stemmed only from private information about the agents' preferences. We now examine the case where agents can also misrepresent their own production possibilities.[7] We impose the following feasibility condition on the reports of the agents' production possibilities: each agent must be 'solvent' (see Shin and Suh [18]); i.e., no agent can exaggerate her production possibilities, or else such a lie would easily be revealed by asking the agent to produce more than she actually can. We show with a 3-person example that ξ^{ϕ^*} is vulnerable to such misrepresentation.

For ease of exposition we will assume that agents have linear preferences with $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 < 1$ the respective slope of their indifference curves in the (x, y)-plane. Because $\lambda_1 > \lambda_2 > \lambda_3$, agent 1 will leave the procedure first, followed by agent 2 and, later, by agent 3. Suppose the production function left over by agent 1 when leaving the procedure is $\bar{f}_1(t) = 2t - \frac{t^2}{2}$. Also, let $f_2(t) = f_3(t) = 3t - \frac{t^2}{2}$. The example is better visualized in the marginal-product space: write $\bar{h}_1(\lambda) = 2 - \lambda$ for $\lambda \in [0, 2]$ and $h_2(\lambda) = h_3(\lambda) = 3 - \lambda$ for

[7] In a different setting, Shin and Suh[18] allowed for misrepresentations of technologies.

$\lambda \in [0,3]$.[8] Recall that once agent 1 has left (i.e. for $\lambda < \lambda_1 = 2$), ξ^{ϕ^*} shares $\bar{h}_1$ according to the ratios of h_2 and h_3. Hence, agent 2's opportunity set is defined by

$$\tilde{h}_2(\lambda) = h_2(\lambda) + \frac{h_2}{h_2 + h_3}(\lambda) \times \bar{h}_1(\lambda) \qquad \text{for } \lambda < 2.$$

The reader can check that agent 2's bundle is then $x_2^* = \frac{5}{2}$ and $y_2^* = \frac{19}{4}$.

If agent 2 reports

$$g_2(\lambda) = \begin{cases} 3\text{-}\lambda & \text{if } \lambda \in [0,1] \\ \frac{5-\lambda}{2} & \text{if } \lambda \in [1,2] \\ \frac{3-\lambda}{2} & \text{if } \lambda \in (2,3] \end{cases}$$

instead of h_2, her bundle is now $x_2' = \frac{5}{2}$ and $y_2' = \frac{20}{27}\ln\left(\frac{5}{8}\right) + \frac{55}{9} \approx 5.763 > y_2^*$ (left to the reader). Clearly, (x_2', y_2') is preferred to (x_2^*, y_2^*).

Note that this example can easily be modified so that preferences (resp. production functions) be made strictly convex (resp. strictly concave).

Remark 3 *Another type of misreport is one where agents downplay their marginal product, f_i', instead of f_i. It is clear from the construction of ϕ^* that a misreport of that sort cannot be beneficial.*

7 Concluding comment

This work contributes to the large literature on the sharing of a single technology between a finite number of agents. This topic was mostly examined in the cost sharing context. Here, we considered the surplus sharing representation as it seemed more relevant to the question of technology pooling. Yet, one can easily transpose our results to the context of cost sharing: if c_i and f_i are dual representations of the same technology ($c_i(y_i) = x_i \iff y_i = f_i(x_i)$), it will also be true of the corresponding aggregate functions ($C^*(y) = x \iff y = F^*(x)$). The optimal path, ϕ^*, that emerged from this paper has a natural cost-sharing counterpart, ψ^*, given by

$$\psi_i^*(y) = f_i(\phi_i^*(x)) \qquad \text{where } x = C^*(y).$$

[8] These functions are obtained by the transformation $h_i(\lambda) = \left(f_i'\right)^{-1}(\lambda)$.

A Proofs

A.1 Proof of Theorem 2

Notation: We say that a matrix $[\alpha_{i,j}]$ has a cycle $(i_1, i_2, i_3, ..., i_L)$ if the i_k's form a non-repeating sequence with $\alpha_{i_L,i_1} \neq 0$ and $\alpha_{i_k,i_{k+1}} \neq 0$ for all $1 \leq k \leq L-1$. A matrix which has no cycles of length greater than 1 is called acyclic. A square acyclic matrix must have an element j such that $\alpha_{i,j} = 0$ for all $i \neq j$; we call such an element a *tail* element. Fix $F \in \mathcal{F}^c$, we say that a mechanism $\xi \in \mathcal{S}_F^c$ is acyclic at a point $x \in \mathbb{R}_+^N$ if the Jacobian matrix of ξ, $\frac{\partial \xi_i}{\partial x_j}$, is acyclic at that point. Notice that if element j is a tail element of the Jacobian matrix of ξ at a point x, then

$$\frac{\partial \xi_j}{\partial x_j}(x) = F'(|x|). \tag{6}$$

We are given an F-sharing rule ξ in $\mathcal{S}_F^c$ satisfying one of the properties (i) or (ii) in the statement of Theorem 2. We will show that ξ must be a ϕ-rule.

We start the proof by restating two lemmas from the proof of Theorem 2 in [11]. Their proofs still hold in our setting, and we will only state these lemmas.

Lemma 1 *(Lemma 5 in [11]) $n \in \mathbb{N}$. Consider $\xi \in \mathcal{S}_F^c$. If every Nash equilibrium is a strong equilibrium, then ξ is acyclic at all $x \in \mathbb{R}_+^N$.*

Lemma 2 *(Lemma 6 in [11]) $n \in \mathbb{N}$. Consider $\xi \in \mathcal{S}_F^c$. If there is at most one Nash equilibrium of the game $G(F, \xi; u)$ for every profile $u \in \mathcal{U}^N$, then ξ is acyclic at all $x \in \mathbb{R}_+^N$.*

The heart of the proof consists in establishing the following lemma.

Lemma 3 *$n = 2$. Consider an F-sharing rule $\xi \in \mathcal{S}_F^c$. Such a rule is a ϕ-rule if and only if the matrix $\frac{\partial \xi_i}{\partial x_j}$ is acyclic for all $x \in \mathbb{R}_+^2$.*

The "only if" part follows directly from the defining formula of ϕ-rules, where it is clear that $\frac{\partial \xi_i}{\partial x_j}(x) = 0$ if and only if $\delta_j(x_j) \geq \delta_i(x_i)$ and $i \neq j$.

Now to the proof of the "if" part. For notational simplicity, we define SW (resp. NE) to be the subset of $\mathbb{R}_+^2$ where element 1 (resp. element 2) is a tail element of the Jacobian matrix of ξ: $\frac{\partial \xi_2}{\partial x_1} = 0$ (resp. $\frac{\partial \xi_1}{\partial x_2} = 0$). We write $D = SW \cap NE$, D is the subset of $\mathbb{R}_+^2$ on which the matrix $\frac{\partial \xi_i}{\partial x_j}$ is diagonal. We define also $SW^* = SW \backslash D$ and $NE^* = NE \backslash D$; by continuity of the partial derivatives of ξ and acyclicity, SW^* and NE^* are open in $\mathbb{R}_+^2$ while SW and NE are closed[9].

The rest of the proof is divided into six steps. We show that the set D is the image of a path $\phi \in \mathcal{P}^c$ and deduce that ξ must be the fixed path method generated by ϕ. The statements of most steps will consist of two symmetrical statements (one per agent), we shall only establish one of them as the other follows by symmetry.

Step 1 **(i)** $\mathbb{R}_+ \times \{0\} \subseteq SW$ and $\{0\} \times \mathbb{R}_+ \subseteq NE$.

(ii) D is nonempty and closed.

(iii) Let $a = (a_1, a_2) \in \mathbb{R}_+^2$, then

$$\begin{aligned} a &\in SW^* \implies (a_1 + \lambda, a_2) \in SW^* \quad \text{for any } \lambda \geq 0, \text{ and} \\ a &\in NE^* \implies (a_1, a_2 + \lambda) \in NE^* \quad \text{for any } \lambda \geq 0. \end{aligned}$$

(i). From ZOZI: $\xi_2(x_1, 0) = 0$ for any $x_1 \geq 0$, therefore $\frac{\partial \xi_2}{\partial x_1}(x_1, 0) = 0$ for any $x_1 \geq 0$. **(ii)**. The non-emptiness of D follows from continuity of the partial derivatives and acyclicity: any continuous curve joining the vertical axis ($\subseteq NE$) to the horizontal axis ($\subseteq SW$) must contain a point in D. Also, D is closed as the intersection of two closed sets.

(iii). Let $a = (a_1, a_2) \in SW^*$ and assume there exists $\bar{a}_1 > a_1$ such that $(\bar{a}_1, a_2) \in NE$. Because SW^* is open, let $]a_1^-, a_1^+[$ be the largest interval containing a_1 on which $(x_1, a_2) \in SW^*$; note that it is non-empty. Because $(0, a_2) \in NE$ (from *(i)*) and $(\bar{a}_1, a_2) \in NE$, it follows that $0 \leq a_1^- < a_1^+ \leq \bar{a}_1$. Also, by continuity of the partials of ξ, $(a_1^-, a_2) \in D$ and $(a_1^+, a_2) \in D$.

SW^* being open, there exists a neighborhood of $\{(x_1, a_2) | x_1 \in]a_1^-, a_1^+[\}$ which is included in SW^*. On this neighborhood, $\frac{\partial \xi_2}{\partial x_1} = 0$; i.e., ξ_2 is independent of x_1. Therefore, the expression $\frac{\xi_2(x_1, x_2+h) - \xi_2(x_1, x_2)}{h}$ is also independent of x_1 on this neighborhood; therefore $\frac{\partial \xi_2}{\partial x_2}$ is independent of x_1 on $\{(x_1, a_2) | x_1 \in]a_1^-, a_1^+[\}$. Hence $\frac{\partial \xi_2}{\partial x_2}(a_1^-, a_2) = \frac{\partial \xi_2}{\partial x_2}(a_1^+, a_2)$. Because $j = 2$ is a tail element of the Jacobian matrix of ξ at (a_1^-, a_2) and (a_1^+, a_2), it follows from (6) that $\frac{\partial \xi_2}{\partial x_2}(a_1^-, a_2) = F'(a_1^- + a_2)$ and $\frac{\partial \xi_2}{\partial x_2}(a_1^+, a_2) = F'(a_1^+ + a_2)$. Hence, $F'(a_1^- + a_2) = F'(a_1^+ + a_2)$ contradicting the strict concavity of F.

We introduce some terminology. We say that a subset $A \subseteq \mathbb{R}^2$ is NW-comprehensive (resp. SE-comprehensive) if $\mathbb{R}_+^2 \cap (A + \mathbb{R}_- \times \mathbb{R}_+) \subseteq A$ (resp. $\mathbb{R}_+^2 \cap (A + \mathbb{R}_+ \times \mathbb{R}_-) \subseteq A$).

[9] Closedness and openness are defined in the relative topology on $\mathbb{R}_+$.

Step 2 SW^* and SW are SE-comprehensive; NE^* and NE are NW-comprehensive.

Let $a = (a_1, a_2) \in SW^*$ and $x = (x_1, x_2)$ such that $x_1 \geq a_1$ and $x_2 \leq a_2$. If $(x_1, x_2) \in NE^*$, then we would have $(x_1, a_2) \in SW^* \cap NE^*$ from the previous step, which is clearly impossible. Hence $x \in SW$. Therefore ξ_2 is independent of x_1 in the region south-east to a. It follows again (see Step 1) that $\frac{\partial \xi_2}{\partial x_2}$ is also independent of x_1 on that domain.

Assume there exists $b = (b_1, b_2) \in NE$ with $b_1 \geq a_1$ and $b_2 \leq a_2$. The case $b_2 = a_2$ has been covered in the previous step, so we will assume $b_2 < a_2$ from now on. Assume $b_1 > a_1$. From the preceding paragraph, $b \in SW$, therefore $b \in D$. However, note that it follows from Step 1, (iii), that $(x_1, b_2) \in NE$ for any $x_1 \in [a_1, b_1]$; hence, $(x_1, b_2) \in NE \cap SW = D$ for any $x_1 \in [a_1, b_1]$. In particular, $(a_1, b_2) \in D$. Therefore, from the previous paragraph: $\frac{\partial \xi_2}{\partial x_2}(a_1, b_2) = \frac{\partial \xi_2}{\partial x_2}(b_1, b_2)$. By (6), $F'(a_1 + b_2) = F'(b_1 + b_2)$, contradicting the strict concavity of F. If $b_1 = a_1$, the result follows from the openness of SW^*: there exists $\varepsilon > 0$ such that $(x_1, a_2) \in SW^*$ for any $x_1 \in [a_1 - \varepsilon, a_1]$. We repeat the previous argument.

We proved that SW^* is SE-comprehensive, a direct consequence is the NW-comprehensiveness of NE. The rest of the claim can be proved symmetrically.

Step 3 For any $(a_1, a_2) \in D$,

$$\begin{aligned} a_1 = 0 &\implies (0, x_2) \in D \quad \forall x_2 \in [0, a_2] \\ a_2 = 0 &\implies (x_1, 0) \in D \quad \forall x_1 \in [0, a_1] \end{aligned} \tag{5.a}$$

$$\begin{aligned} a_2 > 0 &\implies (x_1, a_2) \in SW^* \quad \forall x_1 > a_1 \\ a_1 > 0 &\implies (a_1, x_2) \in NE^* \quad \forall x_2 > a_2 \end{aligned} \tag{5.b}$$

$$\begin{aligned} a_1 > 0 &\implies (a_1, x_2) \notin D \quad \forall x_2 \neq a_2 \\ a_2 > 0 &\implies (x_1, a_2) \notin D \quad \forall x_1 \neq a_1 \end{aligned} \tag{5.c}$$

(5.a). Assume $(0, a_2) \in D$ and let $x_2 \in [0, a_2]$. From Step 1 (i), $(0, x_2) \in NE$, and from Step 1 (iii) it must be that $(0, x_2) \notin NE^*$.

(5.b). Assume $a_2 > 0$ and assume there exists $b_1 > a_1$ such that $(b_1, a_2) \in NE$; then by Step 2, $(b_1, a_2) \in D$. It follows from the SE-comprehensiveness of SW that for all $x_2 < a_2$ and all $x_1 \in [a_1, b_1]$, $(x_1, x_2) \in SW$. On that domain, ξ_2 is independent of x_1, therefore $\frac{\partial \xi_2}{\partial x_2}$ is independent of x_1 also. It follows that $\frac{\partial \xi_2}{\partial x_2}(a_1, a_2) = \frac{\partial \xi_2}{\partial x_2}(b_1, a_2)$, which implies $F'(a_1 + a_2) = F'(b_1 + a_2)$, in contradiction with the strict concavity of F.

(5.c). Assume $a_1 > 0$. We only need to check $(a_1, x_2) \notin D$ for any $x_2 < a_2$ as the case $x_2 > a_2$ is covered by (5.b). Assume there exists $b_2 \in [0, a_2[$ such that $(a_1, b_2) \in D$. Because $a_1 > 0$, applying (5.b) to (a_1, b_2) yields $(a_1, a_2) \in NE^*$, a contradiction. Therefore (5.c) holds.

Step 4 $\exists i \in \{1, 2\}\ \forall x_i \geq 0\ \exists x_j \geq 0$ with $j \neq i$ such that $(x_1, x_2) \in D$.

Assume the statement is not true. Then, there exists $x_1 \geq 0$ such that for any $\lambda \geq 0$, $(x_1, \lambda) \notin D$. From Step 1, $(x_1, 0) \in SW$, therefore by smoothness and acyclicity of ξ, it must be that $(x_1, \lambda) \in SW$ for all $\lambda \geq 0$. Similarly, there exists $x_2 \geq 0$ such that $(\lambda, x_2) \in NE$ for any $\lambda \geq 0$. Hence, $(x_1, x_2) \in SW \cap NE = D$, contradicting our assumption.

Without loss of generality, we will assume for the rest of the proof that for any $x_1 \geq 0$ there exists $x_2 \geq 0$ such that $(x_1, x_2) \in D$. From Step 3, x_2 is unique for any $x_1 > 0$.

Step 5 D is the graph of a continuous increasing path of $\mathbb{R}^2_+$.

Define $P(x_1) = \max \{x_2 \in \mathbb{R}_+ | (x_1, x_2) \in D\}$ for all $x_1 \geq 0$. It follows from Step 2 that P is non-decreasing and strictly increasing on $P^{-1}(\mathbb{R}_{++})$. Also, the graph of the restriction of P to $\mathbb{R}_{++}$ is $D \cap]0, +\infty[\times \mathbb{R}_+$. Because the latter set is closed in $]0, +\infty[\times \mathbb{R}_+$ (as the intersection of SW and NE), P is continuous on $\mathbb{R}_{++}$.

Define $l_2 = \lim_{x_1 \downarrow 0} P(x_1)$; we claim that $l_2 = P(0)$. By closedness of D, $(0, l_2) \in D$. It follows that $l_2 \leq P(0)$ otherwise the very definition of P would be contradicted. Now, if $P(0) > 0$, we show that $l_2 \geq P(0)$. If $l_2 < P(0)$, by continuity of P on $\mathbb{R}_{++}$ there exists $x_1 > 0$ such that $P(x_1) < P(0)$, contradicting the fact that P is non-decreasing. Hence $l_2 = P(0)$ and P is continuous at zero.

Therefore P is continuous on $\mathbb{R}_+$. It follows that

$$D = \{(0, x_2) | x_2 \in [0, P(0)]\} \cup \{(x_1, P(x_1) | x_1 \geq 0\}.$$

Define the function

$$\gamma : \quad t \mapsto \begin{cases} (0, t) & \text{if } t \leq P(0), \\ (\lambda, P(\lambda)) \text{ s.t. } \lambda + P(\lambda) = t & \text{otherwise.} \end{cases}$$

and write $\phi(t) = \gamma(t) \wedge (M_1, M_2)$ for all $t \geq 0$. By continuity and strict monotonicity of P, ϕ is a well-defined path .

Step 6 ξ is the single-path method generated by ϕ.

At any point $x = (x_1, x_2)$ on D, it follows from the definition of D that $\frac{\partial \xi_1}{\partial x_1}(x) = \frac{\partial \xi_2}{\partial x_2}(x) = F'(|x|)$. Thus, for any point $x = (x_1, x_2) \in D$, taking the integral along ϕ from the origin to x yields:

$$\xi_i(x) = \int_0^{\delta_i(x_i)} \frac{\partial \xi_1}{\partial x_1}(\phi(t)) d\phi_i(t) = \int_0^{\delta_i(x_i)} F'(t) d\phi_i(t).$$

Now that ξ is defined on D, it can easily be extended to all of $[0, M_1] \times [0, M_2]$ upon noticing that one agent receives all of the surplus after leaving D: it is the unique tail element of the Jacobian matrix of ξ at x. I.e., for any $x = (x_1, x_2) \in [0, M_1] \times [0, M_2]$ (and, without loss, we assume $\delta_1(x_1) \leq \delta_2(x_2)$)

$$\xi_1(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_1(t) \quad \text{and} \quad \xi_2(x) = \int_0^{\delta_2(x_2)} F'(t) d\phi_2(t) + F(x_1 + x_2) - F(\delta_1(x_1),$$

completing the proof of Theorem 2.

A.2 Proof of Theorem 3

Before proving Theorem 3, we present a lemma establishing that under any fixed path method, ξ^ϕ, any positive level of output, x_i, can be guaranteed at equilibrium by some preference u_i^* for agent i. Its proof can be found in Appendix A.3.

Lemma 4 *Let $\phi \in \mathcal{P}$, $i \in N$. For any $x_i > 0$, there exists a preference $u_i^* \in \mathcal{U}$ such that the following holds:*

$$\forall u_{-i} \in \mathcal{U}^{N \setminus i} \quad x_i^* = x_i;$$

where x^* denotes the unique Nash equilibrium of $G(\xi^\phi; u_i^*, u_{-i})$.

Now to the proof of Theorem 3. Let $\phi \in \mathcal{P}$ such that ξ^ϕ pools f. For the rest of the proof we will write F instead of F^* as no confusion is possible.

Fix $x \in \times_i [0, M_i]$ such that $\delta_i^*(x_i) = \delta_j^*(x_j)$ for all $i, j \in N$; i.e. x is a point on the graph of ϕ^*. From Lemma 4, there exists a preference profile $u \in \mathcal{U}^N$ such that x is the unique Nash equilibrium of $G(\xi^\phi; u)$. It follows that ξ^ϕ pools f only if for any $i \in N$ and any $x_i > 0$ the following holds:

$$\int_0^{\delta_i(x_i)} F'(t) d\phi_i(t) \geq \int_0^{x_i} f_i'(t) dt.$$

By (5) and the definitions of δ_i and δ_i^*, this transforms into

$$\int_0^{x_i} F'(\delta_i(t))dt \geq \int_0^{x_i} F'(\delta_i^*(t))dt \tag{7}$$

for all $i \in N$ and all $x_i > 0$. Let $i \in N$ and define $H_i(x_i) = \int_0^{x_i} F'(\delta_i(t))dt$ for any $x_i \geq 0$; H_i is strictly increasing and strictly concave. Hence,

$$\begin{aligned} H_i(x_i) &\leq H_i(\phi_i \circ \delta_i^*(x_i)) + H_i'(\phi_i \circ \delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \\ \text{i.e.} \quad H_i(x_i) &\leq H_i(\phi_i \circ \delta_i^*(x_i)) + F'(\delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \end{aligned} \tag{8}$$

with equality if and only if $x_i = \phi_i \circ \delta_i^*(x_i)$. It follows from equations (7) and (8) that

$$\begin{aligned} & \int_0^{x_i} F'(\delta_i^*(t))dt \leq \int_0^{\phi_i \circ \delta_i^*(x_i)} F'(\delta_i(t))dt + F'(\delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \\ \iff & \int_0^{x_i} F'(\delta_i^*(t))dt \leq \int_0^{\delta_i^*(x_i)} F'(t)d\phi_i(t) + F'(\delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \\ \iff & \int_0^{x_i} F'(\delta_i^*(t))dt \leq -\int_0^{\delta_i^*(x_i)} \phi_i(t)F''(t)du + F'(\delta_i^*(x_i)) \cdot x_i \end{aligned}$$

the last expression is obtained by integrating by parts. Rearranging yields:

$$\int_0^{\delta_i^*(x_i)} \phi_i(t)F''(t)dt \leq F'(\delta_i^*(x_i)) \cdot x_i - \int_0^{x_i} F'(\delta_i^*(t))dt.$$

Recall that $\delta_i^*(x_i) = \delta_j^*(x_j)$ for all $i \in N$; and write $z = \delta_i^*(x_i)$ for any i. Summing up over all $i \in N$ and using the fact that $\sum_i \phi_i(t) = t$ for any $t \geq 0$ and $\sum_i x_i = \sum \phi_i^*(z) = z$, we get:

$$\begin{aligned} & \int_0^z tF''(t)dt \leq F'(z) \cdot z - \sum_{i=1}^n \int_0^{\phi_i^*(z)} F'(\delta_i^*(t))dt \\ \iff & \int_0^z tF''(t)dt \leq F'(z) \cdot z - \sum_{i=1}^n \int_0^z F'(t)d\phi_i^*(t) \end{aligned}$$

From $\sum_i \phi_i^*(t) = t$ and integrating by parts again, this yields an equality. Therefore, equation (7) must be an equality for all $i \in N$. The choice of j and x_j being arbitrary, it follows that $\delta_i(x_i) = \delta_i^*(x_i)$ for all $x_i \in [0, M_i]$ and for all $i \in N$. That is to say that $\phi_i \equiv \phi_i^*$ for all $i \in N$, proving the theorem.

Remark 4 *In the definition of an f-pooling method, we could replace the voluntary participation requirement with the following weaker one and Theorem 3 would still hold:*

for any profile $u \in \mathcal{U}^N$ and any Nash equilibrium x^* of $G(\xi; u)$ the following *holds:*

$$\xi_i(x^*) \geq f_i(x_i^*) \qquad \forall i \in N.$$

From the strict monotonicity of preferences, this requirement is clearly weaker than expression (4).

A.3 Proof of Lemma 4

Notation: We fix a production function $F \in \mathcal{F}$, a path $\phi \in \mathcal{P}$ and a preference profile $u \in \mathcal{U}^N$. As no confusion may arise, we shall write ξ instead of ξ^ϕ. We denote by F'_- (resp. F'_+) the left (resp. right) derivative of F. Similarly, $\frac{\partial^-}{\partial\lambda}$ (resp. $\frac{\partial^+}{\partial\lambda}$) is the left-derivative (resp. right-derivative) operator. Also, we write:

(i) $\delta(x_1, ..., x_n) = (\delta_1(x_1), \delta_2(x_2), ..., \delta_n(x_n))$ for any $x \in \times_{i\in N}[0, M_i]$,

(ii) $(t_1, t_2, ..., t_{i-1}, t_i \cdot (n-i))$ is the vector of $\mathbb{R}^N_+$ with the last $(n-i)$ coordinates equal to t_i,

(iii) for any $(t_1, ..., t_n) \in \mathbb{R}^N_+$, $\phi(t_1, ..., t_n) = (\phi_1(t_1), \phi_2(t_2), ..., \phi_n(t_n))$ with a slight abuse of notation.

Let $i \in N$ and $x_i > 0$. Consider a preference (utility) u_i^* which is quasi-linear with respect to y_i such that its indifference curves are piecewise linear with a single kink at (x_i, y_i) for any $y_i \in \mathbb{R}$. Set the slope of these indifference curves to be no greater than $F'_-(\delta_i(x_i))$ before x_i and no smaller than $F'_+(x_i)$ after x_i; where "before x_i" (resp. "after x_i") stands for "at any point of $\mathbb{R}_+ \times \mathbb{R}$ with first coordinate smaller (resp. greater) than x_i".

We show below that the former quantity is the smallest variation in output that agent i can obtain via ξ by deviating infinitesimally from x_i: it corresponds to the case where she is the first one served along the path (i.e., the agent with smallest $\delta_j(x_j)$). On the other hand, $F'_+(x_i)$ is the largest variation in output obtainable via ξ at x_i by deviating marginally from x_i; it corresponds to the case where she receives all the output up to $F(x_i)$ ($\delta_j(x_j) = 0$ for all $j \neq i$).

Indeed, let $x_{-i} \in \mathbb{R}^{N\setminus i}_+$; then, from the definition of ξ, and keeping in mind that $|\cdot|$ returns the sum of the coordinates of a vector and $\wedge$ is the componentwise minimum of two vectors,

$$\frac{\partial^-}{\partial\lambda}\xi_i(\lambda, x_{-i}) = F'_-\left(|(\lambda, x_{-i}) \wedge \phi\left(\delta_i(\lambda) \cdot n\right)|\right) \quad \text{and} \quad \frac{\partial^+}{\partial\lambda}\xi_i(\lambda, x_{-i}) = F'_+\left(|(\lambda, x_{-i}) \wedge \phi\left(\delta_i(\lambda) \cdot n\right)|\right) \ .$$

As the ith component of both vectors x and $\phi(\delta_i(x_i)\cdot n)$ is equal to x_i, the concavity of F yields $F'_+(|x \wedge \phi(\delta_i(x_i)\cdot n)|) \leq F'_+(x_i)$. Moreover, the concavity of F also yields $F'_-(|x \wedge \phi(\delta_i(x_i)\cdot n)|) \geq F'_-(|\phi(\delta_i(x_i)\cdot n)|)$; notice that this last term equals $F'_-(\delta_i(x_i))$. It follows from these two facts that:

$$\frac{\partial^-}{\partial\lambda}\xi_i(\lambda, x_{-i})\bigg|_{\lambda=x_i} \geq F'_-(\delta_i(x_i)) \quad \text{and} \quad \frac{\partial^+}{\partial\lambda}\xi_i(\lambda, x_{-i})\bigg|_{\lambda=x_i} \leq F'_+(x_i)\ .$$

Hence, for any $x_{-i} \in \mathbb{R}_+^{N\setminus i}$, the slope of $\xi_i(\lambda, x_{-i})$ at $\lambda = x_i$ lies between $F'_-(\delta_i(x_i))$ and $F'_+(x_i)$. It follows from the strict concavity of $\xi_i(\cdot, x_{-i})$ that x_i maximizes $u_i^*(\lambda, \xi_i(\lambda, x_{-i}))$ on $\mathbb{R}_+^{N\setminus i}$ for any $x_{-i} \in \mathbb{R}_+^{N\setminus i}$, completing the proof of the lemma.

References

[1] S. Barberà, M. Jackson, Strategy-Proof Exchange, *Econometrica* **63** (1995), 51-87.

[2] L.C. Corchón, M. S. Puy, Existence and Nash implementation of efficient sharing rules for a commonly owned technology, *Soc. Choice Welfare* **19** (2002), 369-379.

[3] B. Craig, J. Pencavel, Participation and Productivity: A comparison of worker cooperatives and conventional firms in the plywood industry", *Brookings Pap. Econ. Act. Microeconomics.* **vol. 1995** (1995), 121-174.

[4] P. Dasgupta, P. Hammond, E. Maskin, The implementation of social choice rules: Some results on incentive compatibility, *Rev. Econ. Stud.* **46** (1979), 185-216.

[5] E.J. Friedman, Strategic properties of heterogeneous serial cost sharing, *Math. Soc. Sci.* **44** (2002), 145-154.

[6] E.J. Friedman, Strong monotonicity in surplus sharing, *Econ. Theory* **23** (2004), 643-658.

[7] M.A. de Frutos, Decreasing Serial Cost Sharing under Economies of Scale, *J. Econ. Theory* **79** (1998), 245-275.

[8] D. Israelsen, Collectives, Communes, and Incentives, *J. Compar. Econ.* **4** (1980), 99-124.

[9] J. Leroux, Strategy-proofness and efficiency are incompatible in production economies, *Economics Letters*, 2004, forthcoming.

[10] H. Moulin, Incremental cost sharing: Characterization by coalition strategy-proofness, *Soc. Choice Welfare* **16** (1999), 279-320.

[11] H. Moulin, S. Shenker, Serial Cost Sharing, *Econometrica* **60** (1992), 1009-1037.

[12] T. Saijo, Incentive compatibility and individual rationality in public good economies, *J. Econ. Theory* **55** (1991), 203-212.

[13] M.A. Satterthwaite, H. Sonnenschein, Strategy-Proof Allocations at Differentiable Points, *Rev. Econ. Stud.* **48** (1981), 587-597.

[14] A.K. Sen, Labour allocation in a cooperative enterprise, *Rev. Econ. Stud.* **33** (1966), 361-371.

[15] Serizawa, S. (2002), "Inefficiency of Strategy-Proof Rules for Pure Exchange Economies", *Journal of Economic Theory*, **106**, 219-241.

[16] S. Shenker, On the strategy-proof and smooth allocation of private goods in a production economy, Mimeo, Xerox PARC (1992).

[17] S. Shin, S.-C. Suh, Double implementation by a simple game form in the commons problem, *J. Econ. Theory* **77** (1997), 205-213.

[18] S. Shin, S.-C. Suh, Strategy-proofness and efficiency in a simple production economy, *Soc. Choice Welfare* **17** (2000), 523-532.

[19] Sprumont, Y. (1991), "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule", *Econometrica*, **59**(2), 509-519.

[20] Y. Sprumont, Ordinal Cost Sharing, *Journal of Economic Theory* **81** (1998), 126-162.

[21] R.E. Townsend, Fisheries self-governance: corporate or cooperative structures, *Marine Policy* **19** (1995), 39-45.

[22] M. Weitzman, Free access vs private ownership as alternative systems for managing common property, *J. Econ. Theory* **8** (1974), 225-234.