Limitation of Efficiency: Strategy-Proofness and Single-Peaked Preferences with Many Commodities *

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ABSTRACT

In this paper, we study a resource allocation problem of economies with many commodities and single-peaked preferences. It is known that the uniform rule is the unique allocation mechanism satisfying *strategy-proofness*, *Pareto efficiency* and *anonymity*, if the number of good is only *one* and pereferences are single-peaked. (Sprumont [7].) However, if the number of goods is greater than one, the situation drastically changes and a tradeoff between efficiency and strategy-proofness arises. The generalized uniform rule in multiple-commodity settings is still starategy-proof, but not Pareto efficient in general. In this paper, we show that in a class of all strategy-proof mechanisms the generalized uniform rule is a "second best" strategy-proof mechanism in that there is no other strategy-proof mechanism which gives a "better" outcome than the generalized uniform rule in terms of Pareto domination.

1 Introduction

Starting from Sprumont's [7] remarkable paper, resource allocation in economies with single-peaked preferences has been studied by many authors. Sprumont [7] presented a beautiful characterization of a resource allocation mechanism which satisfies the three axioms of strategy-proofness (SP), anonymity (AN), and Pareto optimality (PO). Strategy-proofness means that telling a true preference is dominant strategy for all agents in a game of stating their preferences. Anonymity says the mechanism should be independent of "names" of agents. Pareto optimalty requires any allocations obtained by the mechanism must be Pareto optimal with respect to reported preference relations.

Sprumont proposes a resource allocation mechanism called the *uniform rule*. Under the uniform rule the same amount of a single divisible good is basically alloted to everyone except people whose peaks are small enough if excess demand exists or large enough if excess supply exists. He proved that if preferences are single-peaked, *the uniform rule is the unique allocation mechanism satisfying* SP, AN *and* PO.

Sprumont's theorem essentially depends on the assumption that there is only one commodity. If the number of goods is greater than one, the natural extension of the uniform rule may not sastisfy Pareto optimality. Following Amoros [1], we call this extension the *generalized uniform rule*. Assignment of goods under the generalized uniform rule is done by applying repeatedly the single good uniform rule commodity by commodity.

It is easy to show that the generalized uniform rule sastisfies strategy-proofness. (See Amoros [1].) [1] However, as shown in Example 1.1 below, the rule does not satisfy Pareto optimality.

Example 1.1 There are two agents and two goods. Let K_1 and K_2 be the amounts of goods 1 and 2 to be allocated. A rectangle of Figure 1 is an Edgeworth Box. In Figure 1, p^{*1} and p^{*2} designate the peaks of Mr. 1 and Mr.2's preferences, respectively. The middle point M of a rectangle is an allocation point where the equal amounts of goods are alloted for each good. Since for each good $i = 1, 2$, both of them have peaks greater than $K_i/2$, the generalized uniform rule assigns the equal amounts of goods to both agents. This allocation is given by M. However, if indifference curves of both agents at M can be drawn like Figure 1, both of them may have room for improving their welfare. (For example, A is better than M for both agents.) So, M is not Pareto optimal.

The literature on strategy-proofness in economic environments has uncovered a tension between strategy-proofness and Pareto efficiency. For example, Hurwicz [2] shows that in pure exchange economies, there is no strategy-proof, Pareto-efficient, and individually rational rule if there are two agents and two goods. Zhou [8] proves that in pure exchange economies, there is no strategy-proof, Pareto-efficient, and non-dictatorial rule if the number of agens is two and the number of goods is greater than or equal to two. [2]

Example 1.1 suggests that the same kind of a tradeoff between strategy-proofness and efficiency may exist in economies with single-peaked preferences and many commodities. Indeed, Amoros [1] shows that in economies with single-peaked preferences a strategy

[1] It also satisfies anonymity.

[2] See also Ohseto [4], Serizawa [5], Kato and Ohseto [3] and Serizawa and Weymark [6].

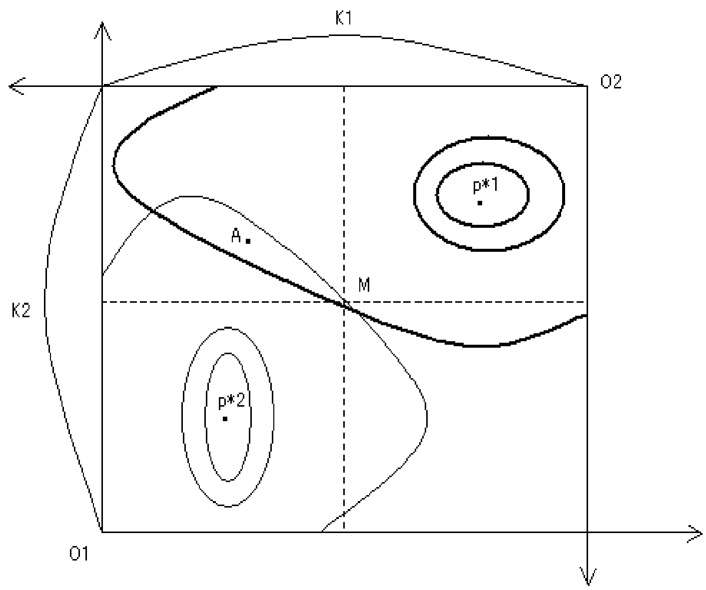

Figure 1:

proof and efficient mechanism must be dictatorial if the number of agents is two and the number of goods is greater than or equal to two. [3] This theorem extends the impossibility theorems of Hurwicz's [2] or Zhou's [8]. One way of escaping this kind of a tradeoff is to drop or replace an axiom. In fact Amoros [1] resolved the difficulty along this line. That is, he replaced efficiency with an axiom of one-sidedness. [4] His main theorem says that if the number of agents is *two* and the number of goods is greater than or equal to two, *the generalized uniform rule is the unique allocation mechanism satisfying envy-freeness, strategy-proofness, and one-sidedness.* [5] Since one-sidedness is a straightforward extension of Sprumont's efficiency concept, [6] Amoros' characterization theorems may be understood as a multiple-good version of Sprumont's characterization.

In the present paper, we study a problem concerning efficiency and strategy-proofness in a multiple-good setup from a much different point of view. Since the tradeoff between

[3] Amoros [1] Theorem 1.

[4] This axiom requires that for each good, the amounts of the good received by everyone is located on the same side of his own peak. That is, for each good, if the quantity of the good received by an agent is greater than or equal to his own peak amount, then the quantities of the good received by the remaining people should be greater than or equal to their own peak amounts, and *vice versa.* If the number of good is one, the one-sidedness is equivalent to Pareto optimality. However, the number of good is greater than one, the one-sidedness does not necessarily imply the Pareto optimality. Amoros [1] called the axiom *Condition E* (CE).

[5] Amoros [1] Theorem 2. In his Theorem 3, he replaces the envy-freeness with the *weak anonymity.*

[6] As noted in footnote 4, one-sidedness is equivalent to Pareto optimality if the number of good is one. In [7], Sprumont assumed Pareto optimalty because some properties of one-sidedness are required in his proof. Hence, in multiple-good economies, only one-sidedness is required for extending Sprumont's characterization.

efficiency and strategy-proofness is inevitable in this setup, we are only interested in a class of strategy-proof mechanisms. Our fundamental question in this research is as follows: is there an "upper bound" on efficiency among all strategy proof mechanisms? Alternatively, what is a strategy-proof rule which cannot be Pareto-dominated by any other strategy-proof rules?

Although the final goal of this research is to find a general upper bound of efficiency in the class of strategy-proof mechanisms, as the first step of the research we will show that such an upper bound does exist. Indeed, the generalized uniform rule is a most efficient strategy-proof mechanism in terms of Pareto domination.

More precisely, our main theorem is that there is no other strategy-proof mechanism whose outcome Pareto-dominates the outcome obtained by the generalized uniform rule.

This paper consists of five sections. In Section 2, the model is constructed and the main theorem is presented. In Section 3, the proof of the main theorem is intuitively described. In Section 4, the proof of the main theorem is given. In Section 5, we conclude the paper by presenting some open questions and conjectures.

2 The Model and the Result

There are 2 agents and m commodities ($m = 1, 2, ...$). Let $N = \{1, 2\}$ be the set of agents and $J = \{1, 2, ..., m\}$ be the set of commodities. For any $i \in J$, let K_i be the quantity of the i-th commodity supplied. Let $K = (K_1, K_2, ..., K_m)$. Let [7]

$$B = \left\{(x^1, x^2) \in R^m_+ \times R^m_+ | x^1_i + x^2_i = K_i \text{ for all } i \in J\right\}.$$

B is the set of all feasible allocations.

Definition 2.1 A preference relation [8] R defined on R^m_+ is *single-peaked*, if there is $p^* \in R^m_+$ such that for all $x, x' \in R^m_+$ with $x \neq x'$, if for all $i \in J$, *either* $p^*_i \geq x_i \geq x'_i$ *or* $x'_i \geq x_i \geq p^*_i$ is true, then xPx', where xPx' means xRx' *but not* $x'Rx$.

Let Γ be the class of all single-peaked preferences on R^m_+. For any $R \in \Gamma$, let $p(R)$ denote the peak of R and for all $i \in J$, $p_i(R)$ is the i-th component of $p(R)$.

Definition 2.2 A *solution* is a mapping $\Psi : \Gamma^2 \to B$.

For all $(R^1, R^2) \in \Gamma^2$ and all $k \in N$, let $\Psi^k(R^1, R^2)$ be the consumption vector alloted to the $k-$th agent in $\Psi(R^1, R^2)$.

Definition 2.3 A solution Ψ is *strategy-proof* (SP), if for all $(R^1, R^2) \in \Gamma^2$,

$$\Psi^1(R^1, \tilde{R}^2) R^1 \Psi(\tilde{R}^1, \tilde{R}^2) \text{ for all } (\tilde{R}^1, \tilde{R}^2) \in \Gamma^2,$$

and

[7] R^m_+ is the set of all $m-$dimensional nonnegative vectors.

[8] A complete, reflexive and transitive binary relation is called a *preference relation*.

$$\Psi^2(\tilde{R}^1, R^2) R^2 \Psi(\tilde{R}^1, \tilde{R}^2) \text{ for all } (\tilde{R}^1, \tilde{R}^2) \in \Gamma^2,$$

Definition 2.4 An allocation $x = (x^1, x^2) \in B$ is *Pareto optimal* for $(R^1, R^2) \in \Gamma^2$, if there is it no $\tilde{x} \in B$ such that

$$\tilde{x}^k R^k x^k \text{ for all } k \in N$$

$$\tilde{x}^{k_0} P^{k_0} x^{k_0} \text{ for some } k_0 \in N$$

Definition 2.5 A solution Ψ is *Pareto optimal* (PO), if for all $R \in \Gamma^2$, the allocation $\Psi(R)$ is Pareto Optimal for $R \in \Gamma^2$.

Definition 2.6 The *generalized uniform rule* is the mapping $U : \Gamma^2 \to B$ that selects for all $R = (R^1, R^2) \in \Gamma^2$ and all $i \in J$, the allocation $x \in B$ such that

(1) if $\sum_{k \in N} p_i(R^k) \geq K_i$, then there exists λ_i such that $x_i^k = \min\left\{p_i(R^k), \lambda_i\right\}$,
(2) if $K_i \geq \sum_{k \in N} p_i(R^k)$, then there exists λ_i such that $x_i^k = \max\left\{p_i(R^k), \lambda_i\right\}$.

The following proposition is owing to Amoros [1] (Proposition 1).

Proposition 2.1 The generalized uniform rule satisfies strategy-proofness.

Our main theorem is the following.

Theorem 2.2 Let Ψ be any strategy-proof solution on Γ^2. If

$$\Psi^k(R) R^k U^k(R) \text{ for all } R = (R^1, R^2) \in \Gamma^2 \text{ and all } k \in N,$$

then $\Psi(R) = U(R)$ for all $R \in \Gamma^2$.

3 Intuitive Explanation of the Proof of the Main Theorem

Although the formal proof of the main theorem is given in the next section, we explain how to prove it intuitively in this section. Consider economies with two people and two goods. To prove the theorem, suppose the contrary. Then, there is a strategy-proof solution $\Psi : \Gamma^2 \to B$ such that

$$\Psi^k(R) R^k U^k(R) \text{ for all } R = (R^1, R^2) \in \Gamma^2 \text{ and for all } k \in N$$

and

$$\Psi(R^*) \neq U(R^*) \text{ for some } R^* = (R^{*1}, R^{*2}) \in \Gamma^2.$$

Let $p^{*1} = p(R^{*1})$ and $p^{*2} = p(R^{*2})$. Let us assume that p^{*1} and p^{*2} are located as in Figure 2. In this case, there are excess demands for both goods and for all $k \in N$ and all

$i \in J$, $p_i^{*k} > K_i/2$. This means that for both agents, their demands are large enough for both commodities. Hence, the generalized uniform rule assigns the point M of Figure 2. M is the equal allocation point for both goods. Let $A = \Psi(R^*)$. Note that $A \neq M$.

We construct a new single-peaked preference relation $\tilde{R}^2$ for Mr.2. Indifference curves of $\tilde{R}^2$ are circles with center A. Consider an economy with preference profile $(R^{*1}, \tilde{R}^2)$. Let us find an allocation assigned by the generalized uniform rule. In the market of the first good, since excess demand exists and both agents have large demand exceeding $K_1/2$, both of them receive equal amounts. In the market for the second good, excess demand still prevails. However, since Mr.2's demand is less than $K_2/2$, the amount of good 2 he receives is equal to his Walrasian demand given by the peak of the preference relation $\tilde{R}^2$. Hence, the allocation assigned by the uniform rule is the point B of Figure 3.

Since $\Psi(R^{*1}, \tilde{R}^2)$ assigns an allocation which is better than or indifferent to B for each agent, $\Psi(R^{*1}, \tilde{R}^2)$ must be located in the shaded area of Figure 3. Since A is the peak of $\tilde{R}^2$, every point in this shaded area is worse than A for Mr.2. Hence, when Mr. 1 reports R^{*1}, Mr. 2 has an incentive to tell a lie according to his preference $\tilde{R}^2$. That is, Mr.2 will be better off in terms of his preference $\tilde{R}^2$, if he reports R^{*2} instead of his true preference $\tilde{R}^2$. This contradicts strategy-proofness.

So far, we have shown that the main theorem is true if peaks of the preference relations R^{*1} and R^{*2} are located as in Figure 2. To complete the proof, we need to check the remaining possibilities for the location of the peaks, but the intuitive idea for the proof is essentially the same. In the next section, we give a formal proof of the theorem in the two-agent and m good case.

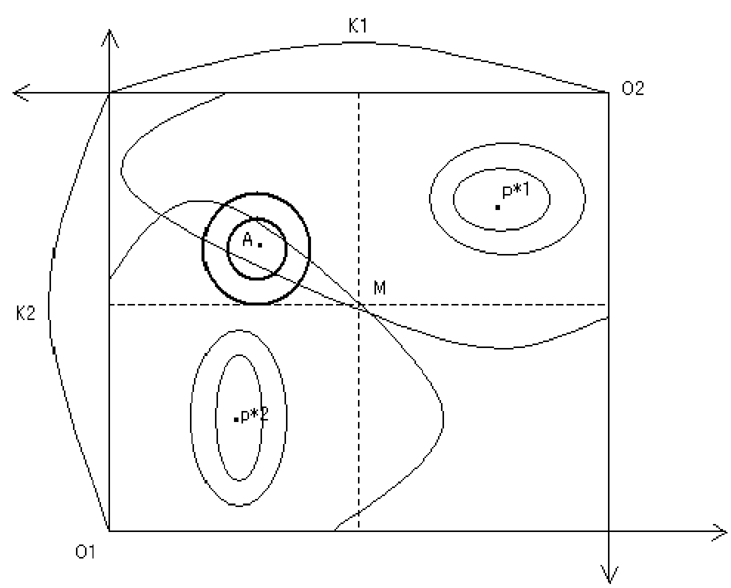

Figure 2:

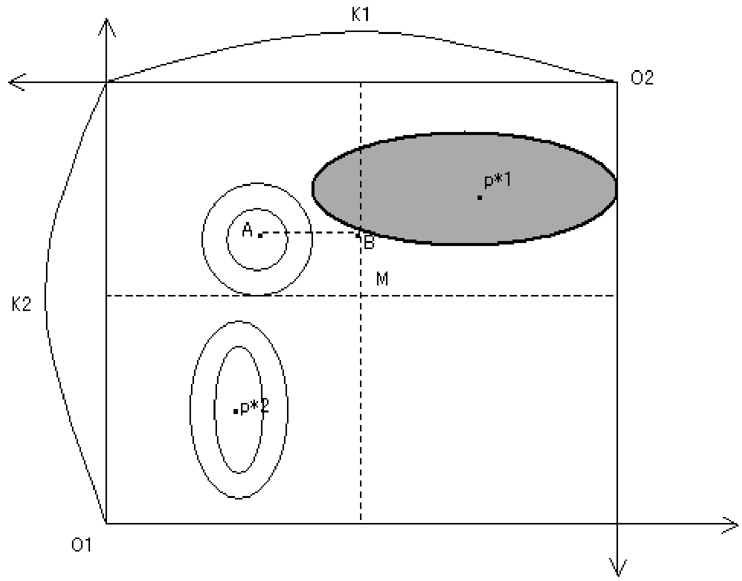

Figure 3:

4 Proof of the Theorem

In this section, we prove Theorem 2.2. We argue by contradiction. Suppose that there is a strategy-proof solution $\Psi : \Gamma^2 \to B$ such that

$$\Psi^k(R)R^kU^k(R) \text{ for all } R = (R^1, R^2) \in \Gamma^2 \text{ and for all } k \in N$$

and

$$\Psi(R^*) \neq U(R^*) \text{ for some } R^* = (R^{*1}, R^{*2}) \in \Gamma^2.$$

Let $x^* = (x^{*1}, x^{*2}) = \Psi(R^*), d = (d^1, d^2) = U(R^*), p^{*1} = p(R^{*1})$ and $p^{*2} = p(R^{*2})$.

Since $x^{*1} + x^{*2} = K = (K_1, K_2, ..., K_m)$, $\Psi(R^*) \neq U(R^*)$ implies that $x^{*1} \neq d^1$ and $x^{*2} \neq d^2$. Since x^* Pareto-dominates d by assumption, $x^{*1}R^{*1}d^1$ and $x^{*2}R^{*2}d^2$ must hold.

Let $\tilde{R}^2 \in \Gamma$ be represented by the utility function $\tilde{u}^2(x^2)$, where

$$\tilde{u}^2(x^2) = -\sum_{i=1}^{m}(x_i^2 - x_i^{*2})^2 \text{ for all } x^2 \in R_+^m$$

Note that $p(\tilde{R}^2) = x^{*2}$. If $m = 2$, the indifference curves of $\tilde{R}^2$ are circles with center x^{*2}.

Let $y^* = (y^{*1}, y^{*2}) = U(R^{*1}, \tilde{R}^2)$.

Lemma 4.1 $y^{*1}R^{*1}x^{*1}$.

Proof: For all $i \in J$, let

$$y^{*1}(i) = \left(y_1^{*1}, y_2^{*1}, ..., y_i^{*1}, x_{i+1}^{*1}, ..., x_m^{*1}\right),$$

and let

$$y^{*1}(0) = x^{*1}.$$

Obviously, $y^{*1}(m) = y^{*1}$.

First, we show that $y^{*1}(1)R^{*1}x^{*1}$. To this end, note that by the definition of the generalized uniform rule, for all $i \in J$,

$$y_i^{*1} = x_i^{*1}, p_i^{*1}, or\ K_i/2.$$

If $y_1^{*1} = x_1^{*1}$, this is obvious because $y^{*1}(1) = x^{*1}$. If $y_1^{*1} = p_1^{*1}$, by single-peakedness, we have

$$y^{*1}(1) = \left(p_1^{*1}, x_2^{*1}, ..., x_m^{*1}\right) R^{*1} \left(x_1^{*1}, x_2^{*1}, ..., x_m^{*1}\right) = x^{*1}.$$

Finally, if $y_1^{*1} = K_1/2$, there are two possibilities: (a) $K_1/2 \geq p_1^{*1}$ and $K_1/2 \geq x_1^{*2}$, and (b) $p_1^{*1} \geq K_1/2$ and $x_1^{*2} \geq K_1/2$.

In each case, either

$$x_1^{*1} \geq y_1^{*1} = y_1^{*1}(1) \geq p_1^{*1} \text{ or } p_1^{*1} \geq y_1^{*1} = y_1^{*1}(1) \geq x_1^{*1}.$$

For all $j \in J$ with $j \neq 1$, either

$$x_j^{*1} = y_j^{*1}(1) \geq p_j^{*1} \text{ or } p_j^{*1} \geq y_j^{*1}(1) = x_j^{*1}.$$

Hence, by the definition of single-peakedness,we have

$$y^{*1}(1)R^{*1}x^{*1}.$$

By repeating the same argument, we show that for all $i = 1, 2, ..., m$,

$$y^{*1}(i)R^{*1}y^{*1}(i-1).$$

Then,

$$y^{*1} = y^{*1}(m)R^{*1}y^{*1}(m-1)R^{*1}.....y^{*1}(1)R^{*1}y^{*1}(0) = x^{*1}.$$

QED

Lemma 4.2 *$x_i^{*1} = y_i^{*1}$ if and only if*

$$(1) \text{ if } x_i^{*1} > K_i/2, \text{ then } p_i^{*1} > K_i/2 \text{ and } p_i^{*1} \geq x_i^{*1},$$

and

$$(2) \text{ if } x_i^{*1} < K_i/2, \text{ then } p_i^{*1} < K_i/2 \text{ and } x_i^{*1} \geq p_i^{*1}.$$

Proof: (a)Necessity. Note that $y_i^{*1} = x_i^{*1}, p_i^{*1}$, or $K_i/2$.
(i) If $x_i^{*1} > K_i/2$, then $x_i^{*2} < K_i/2$.
If $K_i/2 \geq p_i^{*1}$, then $y_i^{*1} = K_i/2$. Therefore, $y_i^{*1} = K_i/2 < x_i^{*1}$, a contradiction. Hence, $p_i^{*1} > K_i/2$. Then, $y_i^{*1} = \min\{p_i^{*1}, x_i^{*1}\}$. Since $y_i^{*1} = x_i^{*1}$, $p_i^{*1} \geq x_i^{*1}$.
(ii) If $x_i^{*1} < K_i/2$, then $x_i^{*2} > K_i/2$. If $p_i^{*1} \geq K_i/2$, then $y_i^{*1} = K_i/2 > x_i^{*1}$, a contradiction. Hence, $p_i^{*1} < K_i/2$. Therefore, $y_i^{*1} = \max\{p_i^{*1}, x_1^{*1}\} = x_i^{*1}$. Then, $x_i^{*1} \geq p_i^{*1}$.

(b) Sufficiency. Assume (1) and (2).
(i) If $x_i^{*1} = K_i/2$, then p_i^{*1} may take any value. Since $x_i^{*2} = K_i/2$, then $y_i^{*1} = K_i/2 = x_i^{*1}$.
(ii) If $x_i^{*1} > K_i/2$, then $x_i^{*2} < K_i/2$ and $p_i^{*1} > K_i/2$. Hence, $y_i^{*1} = \min\{x_i^{*1}, p_i^{*1}\}$. Since $p_i^{*1} \geq x_i^{*1}$, then $y_i^{*1} = x_i^{*1}$.
(iii) If $X_i^{*1} < K_i/2, x_i^{*2} > K_i/2$ and $p_i^{*1} < K_i/2$. Hence, $y_i^{*1} = \max\{x_i^{*1}, p_i^{*1}\}$. Since $x_i^{*1} \geq p_i^{*1}$,then $y_i^{*1} = x_i^{*1}$.

QED

Lemma 4.3 If $x^{*1} = y^{*1}$, then $x^{*1} = d^1$.

Proof: Suppose that $x^{*1} \neq d^{*1}$. For all $i = 1, 2, ..., m$, let

$$d^2(i) = \left(x_1^{*2}, \cdots, x_i^{*2}, d_{i+1}^2, \cdots, d_m^2\right),$$

and let $d^2(0) = d^2$. For $i = 1, 2, ..., m$, we want to show that

$$d^2(i-1) R^{*2} d^2(i).$$

To this end, consider several cases.

CASE 1: $x_i^{*1} = K_i/2$.
(a) If $d_i^2 > K_i/2$, then $d_i^2 = \min\{p_i^{*2}, K_i - p_i^{*1}\}$. Hence,

$$p_i^{*2} \geq d_i^2 > K_i/2 = x_i^{*1} = x_i^{*2}.$$

That is,

$$p_i^{*2} \geq d_i^2(i-1) > d_i^2(i).$$

For all $j \in J$ with $j \neq i$,either

$$p_j^{*2} \geq d_j^2(i-1) = d_j^2(i) \text{ or } d_j^2(i) = d_j^2(i-1) \geq p_j^{*2}.$$

Hence, by the definition of single-peakedness, $d^2(i-1) R^{*2} d^2(i)$.
(b)If $d_i^2 < K_i/2$, then d_i^2= max $\{p_i^{*2}, K_i - p_i^{*1}\}$. Hence,

$$K_i/2 = x_i^{*1} = x_i^{*2} > d_i^2 \geq p_i^{*2}.$$

That is,

$$d_i^2(i) > d_i^2(i-1) \geq p_i^{*2}.$$

For all $j \in J$ with $j \neq i$,either

$$p_j^{*2} \geq d_j^2(i-1) = d_j^2(i) \text{ or } d_j^2(i) = d_j^2(i-1) \geq p_j^{*2}.$$

Hence, by the definition of single-peakedness, $d^2(i-1)R^{*2}d^2(i)$.
(c) If $d_i^2 = K_i/2$, then $d_i^2 = x_i^2$. Hence, $d^2(i-1) = d^2(i)$. Then, $d^2(i-1)R^{*2}d^2(i)$.

CASE 2: $x_i^{*1} > K_i/2$. (i.e., $x_i^{*2} < K_i/2$.)
By Lemma 4.2, $p_i^{*1} > K_i/2$.
If $p_i^{*2} \geq K_i/2$, then $d_i^1 = d_i^2 = K_i/2$. Then,

$$p_i^{*2} \geq d_i^2 = K_i/2 > x_i^{*2}$$

That is,

$$p_i^{*2} \geq d_i^2(i-1) > d_i^2(i).$$

For all $j \in J$ with $j \neq i$,either

$$p_j^{*2} \geq d_j^2(i-1) = d_j^2(i) \text{ or } d_j^2(i) = d_j^2(i-1) \geq p_j^{*2}.$$

Hence, by the definition of single-peakedness, $d^2(i-1)R^{*2}d^2(i)$.
Otherwise (i.e., if $K_i/2 > p_i^{*2}$), $d_i^1 = \min\{p_i^{*1}, K_i - p_i^{*2}\}$, and $d_i^2 = \max\{p_i^{*2}, K_i - p_i^{*1}\}$
(a)If $K_i - p_i^{*1} \geq p_i^{*2}$, then $d_i^2 = K_i - p_i^{*1}$. By Lemma 4.2, $p_i^{*1} \geq x_i^{*1}$. That is, $x_i^{*2} \geq K_i - p_i^{*1}$.
Hence,

$$x_i^{*2} \geq d_i^2 \geq p_i^{*2}$$

That is,

$$d_i^2(i) \geq d_i^2(i-1) \geq p_i^{*2}.$$

For all $j \in J$ with $j \neq i$,either

$$p_j^{*2} \geq d_j^2(i-1) = d_j^2(i) \text{ or } d_j^2(i) = d_j^2(i-1) \geq p_j^{*2}.$$

Hence, by the definition of single-peakedness, $d^2(i-1)R^{*2}d^2(i)$.
(b)If $p_i^{*2} > K_i - p_i^{*1}$, then $d_i^{*2} = p_i^{*2}$. Since $d_j^2(i) = d_j^2(i-1)$ for all $j \in J$ with $j \neq i$, $d_i^2(i-1) = p_i^{*2}(i)$ implies that $d^2(i-1)R^{*2}d^{*2}(i)$ by the definition of single-peakedness.

CASE 3: $x_i^{*1} < K_i/2$. (i.e., $x_i^{*2} > K_i/2$.)
By Lemma 4.2, $p_i^{*1} < K_i/2$.
If $K_i/2 \geq p_i^{*2}$, then $d_i^1 = d_i^2 = K_i/2$. Then,

$$x_i^{*2} > d_i^2 = K_i/2 \geq p_i^{*2}.$$

That is,

$$d_i^2(i) > d_i^2(i-1) \geq p_i^{*2}.$$

For all $j \in J$ with $j \neq i$,either

$$p_j^{*2} \geq d_j^2(i-1) = d_j^2(i) \text{ or } d_j^2(i) = d_j^2(i-1) \geq p_j^{*2}.$$

Hence, by the definition of single-peakedness, $d^2(i-1)R^{*2}d^2(i)$.

Otherwise (i.e., if $K_i/2 < p_i^{*2}$), $d_i^1 = \max\{p_i^{*1}, K_i - p_i^{*2}\}$ and $d_i^2 = \min\{p_i^{*2}, K_i - p_i^{*1}\}$.
(a) If $p_i^{*2} \geq K_i - p_i^{*1}$, then $d_i^2 = K_i - p_i^{*1}$. By Lemma 4.2, $x_i^{*1} \geq p_i^{*1}$. Then, $K_i - p_i^{*1} \geq x_i^{*2}$. Hence,

$$p_i^{*2} \geq d_i^{*2} = K_i - p_i^{*1} \geq x_i^{*2}$$

That is,

$$p_i^{*2} \geq d_i^2(i-1) \geq d_i^2(i).$$

For all $j \in J$ with $j \neq i$,either

$$p_j^{*2} \geq d_j^2(i-1) = d_j^2(i) \text{ or } d_j^2(i) = d_j^2(i-1) \geq p_j^{*2}.$$

Hence, by the definition of single-peakedness, $d^2(i-1)R^{*2}d^2(i)$.
(b) If $p_i^{*2} < K_i - p_i^{*1}, d_i^2 = p_i^{*2}$. Since $d_j^2(i) = d_j^2(i-1)$ for all $j \in J$ with $j \neq i$, $d_i^2(i-1) = p_i^{*2}(i)$ implies that $d^2(i-1)R^{*2}d^2(i)$ by the definition of single=peakedness.

From the results in Cases 1 to 3, we have

$$d^2 = d^2(0)R^{*2}d^2(1)R^{*2}\cdots R^{*2}d^2(m) = x^{*2}.$$

Since $x^{*1} \neq d^1$ and $x^{*1} + x^{*2} = d^1 + d^2 = K, x^{*2} \neq d^2$. Hence, at least one of the above relations must be strict. Therefore, we have

$$d^2 P^{*2} x^{*2}$$

Since $\left(x^{*1}, x^{*2}\right) = \Psi\left(R^{*1}, R^{*2}\right), x^{*2}R^{*2}d^2$, a contradiction.

QED

By Lemma 4.3, $x^{*1} \neq d^1$ implies that $x^{*1} \neq y^{*1}$. Note that in the proof of Lemma 4.1, we proved the relations:

$$y^{*1} = y^{*1}(m)R^{*1}y^{*1}(m-1)R^{*1}\cdots y^{*1}(1)R^{*1}y^{*1}(0) = x^{*1}.$$

Hence, the preference relation between y^{*1} and x^{*1} is strict. That is,

$$y^{*1}P^{*1}x^{*1}$$

This implies that the allocation $\left(x^{*1}, x^{*2}\right)$ cannot be chosen by $\Psi(R^{*1}, \tilde{R^2})$, because $\Psi^1(R^{*1}, \tilde{R^2})R^{*1}y^{*1}$. Since $x^{*2} = p(\tilde{R^2})$, we have $x^{*2}\tilde{P^2}\Psi^2(R^{*1}, \tilde{R^2})$. On the other hand, $x^{*2} = \Psi^2(R^{*1}, R^{*2})$. Hence,

$$\Psi^2(R^{*1}, R^{*2}) = x^{*2}\tilde{P^2}\Psi^2(R^{*1}, \tilde{R^2}).$$

This means that Mr.2 gains by telling a lie if his true preference is $\tilde{R^2}$. This contradicts strategy proofness. This completes the proof of Theorem 2.2.

5 Cocluding Remarks

In this paper we have shown that the generalized uniform rule is a most efficient resource allocation mechanism among all strategy-proof mechanisms. In other words, the generalized uniform rule is a "second best" solution in the class of strategy-proof mechanisms. As a conclusion, we present some open questions and conjectures.

Since Theorem 2.2 was proved for economies with two agents and m goods, to extend this theorem to economies with n agents and m goods is an interesting question.

However, there are some more open questions. That is, is the "converse" of our theorem true? To state this problem, let us introduce a definition. Consider economies with n agents and m goods. Let N be the set of all agents.

Let Λ be a class of solutions defined on Γ^n.

Definition 5.1 For any $\Phi \in \Lambda$, a solution Φ is called to be Λ- *efficient*, when for any mechanism $\Psi \in \Lambda$, if

$$\Psi^k(R) R^k \Phi^k(R) \text{ for all } R \in \Gamma^n \text{ and all } k \in N,$$

then $\Psi(R) = \Phi(R)$ for all $R \in \Gamma^n$.

Using this definition, what we proved in the present paper is that if n=2, then the generalized uniform rule satisfies Λ_{SP}-efficiency, where Λ_{SP} is the set of all strategy-proof mechanisms on Γ^n.

Since the dictatorial mechanism is strategy-proof and Pareto optimal, it also satisfies Λ_{SP}-efficiency.

Using the concept of Λ-efficiency, the "converse" of our main theorem may be stated as follow:

(1) Are the generalized uniform rule and the dictatorial rule only two Λ_{SP}-efficient solutions? Or, are there other Λ_{SP}-fficient solutions?

(2) Is the generalized uniform rule the only one Λ_{SP}-solution satisfying AN? Or, are there any other Λ_{SP}-efficient solutions satisfying AN?

Finally, in Arrow and Debreu's economies with nonsatiated preferences, we may state a similar kind of question. As disucussed in *Introduction,* there are several impossibility theorems insisting that there is a tradeoff between efficiency and strategy-proofness in this kind of settings. [9] If we weaken the requirement of efficiency, we may have the same kind of problems as disucussed in this paper. To investigate whether Λ_{SP} -efficient mechanisms exist in Arrow and Debreu economies would be another interesting question.

[9]Hurwicz [2], Zhou [8], Serizawa [5], Serizawa and Weymark [6], and so on.

References

[1] P. Amoros, Single-Peaked preferences with several commodities, *Soc. Choice Welfare,* 19(2002), 57-67.

[2] L. Hurwicz, On informationally decentralized systems, in C.B.McGuire and R.Radner (eds) *Decision and Organization,* North-Holland, Amsterdam, 1972, 297-336.

[3] M. Kato and S. Ohseto, Toward general impossibility theorems in pure exchange economies, *Soc. Choice Welfare,* 19(2002), 659-664.

[4] S. Ohseto, Strategy-proof and efficient allocation of indivisible good on finitely restricted preference domains, *International J. of Game Theory,* 29(2000), 365-374.

[5] S. Serizawa, Inefficiency of strategy-proof rules for pure exchange economies, forthcoming in *J. Econ. Theory,* 2000.

[6] S. Serizawa and J. Weymark, Efficientf strategy-proof exchange and minimum consumption guarantees, forthcoming in *J. Econ. Theory*, 2002.

[7] Y. Sprumont, The division problem with single-peaked preferences: a characterization of the uniform allocation rule, *Econometrica,* 59(1991), 509-519.

[8] L. Zhou, Inefficiency of strategy-proof allocation mechanisms in pure exchange economies, *Soc. Choice Welfare,* 8(1991), 247-254.