Credit Cycles Redux

Juan Carlos Córdoba — *Rice University*

Marla Ripoll* — *University of Pittsburgh*

September, 2002

Abstract

Theoretical studies have shown that under unorthodox assumptions on preference and production technologies, collateral constraints can act as a powerful amplification and propagation mechanism of exogenous shocks. We investigate whether or not this result holds under more standard assumptions. We find that collateral constraints generate a typically small output amplification. Large amplification is a "knife-edge" type of result.

JEL classification: E32, E44

Keyworkds: collateral constraints, business cycles, heterogeneous agents.

1 Introduction

Business cycle models typically rely on large exogenous shocks to explain fluctuations in aggregate output. This approach is often criticized because shocks of the required magnitude are hard to find in the data (Summers 1986, Cochrane 1994). An alternative explanation is that the economy has some *amplification mechanism* that transforms relatively small shocks into large output fluctuations.

Kiyotaki and Moore (1997) and Kiyotaki (1998) have argued that such mechanism is a particular form of credit-market frictions. Specifically, when debts need to be fully secured by collateral, say land, and the collateral is also an input in production, then a small shock to the economy can be largely amplified. For instance, a small negative shock that reduces the net worth of credit-constrained firms forces them to curtail their investment in land. Land prices and output fall because credit-constrained firms are by nature more productive in the use of land. The fall in the value of the collateral reduces even more the debt capacity of constrained firms, causing additional

*We thank Daniele Coen-Pirani, David DeJong, Jack Ochs, and participants to the Stanford Institute for Theoretical Economics Workshop 2002 for useful comments.

falls in investment, land prices, and output. The cumulative effect could be dramatic, as they show using a carefully designed economy.

The results of Kiyotaki and Moore (1997) (KM henceforth) have launched a significant body of mainly theoretical research. Examples are Krishnamurthy (1998), Kocherlakota (2000), Caballero and Krishnamurthy (2001), and Paasche (2001). However, there has not yet been a systematic assessment of the quantitative significance of collateral constraints as an amplification mechanism. This assessment seems particularly important because theoretical models have used some extreme assumptions in order to boost the amplification. For example, KM introduce enough assumptions to induce constrained agents to fully invest all of the unexpected income; to prevent any response of the interest rate (lenders' preferences are linear); and to enhance the role of collateral in the economy (borrowers' technology is linear in land).[1] Are shocks still significantly amplified under more standard choices of preferences and technologies?

The objective of this paper is to address this question using a simple dynamic general equilibrium model. The model is a two-agent closed economy, in the spirit of KM, but modified to introduce standard specifications of preferences and technologies. In particular, all agents in our economy have concave preferences, have access to concave production technologies, and are required to collateralize their debts. In order to generate productivity gaps between constrained and unconstrained agents, we employ the standard, but nonessential, assumption that agents differ in their discount factors. We constrain the parameters of the economy so that shocks are persistent and the rational expectations equilibrium is unique around the steady state of the model, i.e., the steady state exhibits monotonic saddle-path stability. We use the model to examine the features and parameter values needed in order to achieve large amplification.

The main finding of this paper is that collateral constraints can in fact amplify unexpected shocks to the economy, but the effect is generally small. For the standard values of a capital share of around 0.3, and an elasticity of intertemporal substitution (EIS) of 1, the amplification is close to zero. Large amplification is a "knife-edge" type of result: on the one hand, it occurs at the right

[1]Both the appendix of Kiyotaki and Moore (1997) and Kiyotaki (1998) attempt to relax some of the unorthodox assumptions, but there is no assertion on whether these models can generate large output amplification.

combination of a typically small EIS (below 0.2) and a large share of capital (the collateralizable asset) in the production function. But if the EIS is too small, or the capital share is too large, then the steady state may not be a monotonic saddle path. Instead, the equilibrium may exhibit jagged dynamics, or may not even exist.

To understand why the amplification is typically small, it is useful to break up the response of output to a shock in the following four components:

$$\begin{aligned} \text{output response} \quad = \quad & (\text{productivity gap}) \times (\text{collateral share in production}) \times \\ & (\text{production share constrained agents}) \times (\text{redistribution of collateral})\,. \end{aligned}$$

This expression states that the response of output to shocks is bigger the larger the productivity gap between constrained and unconstrained agents, the larger the share of collateral in the production function, the larger the fraction of output produced by constrained agents, and the larger the redistribution of collateral from unconstrained to constrained agents originated by the shock. Notice that the amplification is caused by the redistribution of collateral from low-productive unconstrained agents to high-productive constrained agents. The expression suggests that the response of output is generally small. For example, if constrained agents are 50% more productive, produce 50% of the total output, and the collateral share is 50%, then constrained agents must increase their holdings of collateral by 800% just to increase output in 1%.

More specifically, there are three main reasons why amplification is typically small. First, the concavity of the production imposes a natural limit on the size of the first three components of the expression above. In that case, the share of collateral is below 1, and there is a trade-off between the productivity gap and the production share: a large productivity gap requires constrained agents to hold a small fraction of the collateral in the economy, which means that their share of the total production must be small. KM avoid this trade-off by assuming that the technology of constrained agents is linear in the collateral.

Second, the concavity of the preferences imposes a natural limit on the size of the fourth component. As constrained agents use the unexpected resources from a positive shock to secure more debt and demand more capital, the interest rate increases to induce unconstrained agents to provide the additional loans. This response of the interest rate limits the magnitude of the redistribution of capital and the response of output to the shock. If preferences are linear, as is the case in KM, then constrained agents can provide the additional loans without any increase in the interest rate. Thus, the asset price effect emphasized by KM is partially offset by the interest rate effect when preferences are concave. We find that for plausible values of the EIS the response of the interest rate almost completely eliminates the asset-price effect.

Finally, concave preferences also limit the size of the fourth component in second way. Consumption smoothing implies that part of the unexpected resources are invested and part are consumed. In KM economy, however, constrained agents invest all the unexpected resources in capital.

The finding that amplification is generally small holds even in the case in which we allow agents to differ not only in their discount factors, but also in the EIS and the capital share in their respective technologies. Overall, our results show that for an empirically plausible calibration, collateral constraints *by themselves* are not enough to account for the large fluctuations of output observed in the data.

Our exercise is similar in spirit to Kocherlakota (2000). He shows that the quantitative significance of the amplification effects generated by endogenous collateral constraints depends crucially on the parameters of the economy, in particular on factor shares. Our paper differs from Kocherlakota's in two ways. First, our economy is closed so that the interest rate is endogenously determined. This allows us to account for general equilibrium effects. Second, as in KM, the distribution of collateral across agents plays a crucial role in our model: this role is lost in Kocherlakota's specification, which eliminates the leverage effect present in KM.

There is a related literature on the importance of financial factors on the investment behavior of firms which emphasizes the role agency costs (see, for example, Bernanke and Gertler, 1989; Bernanke, Gertler and Gilchrist,1999; and Calstrom and Fuerst, 1997 and 2000), and of limited enforceability (see Cooley, Quadrini and Marimon, 2001). These models do not directly incorporate collateral constraints, and consider different mechanisms to the ones analyzed here. We study directly the role of collateral constraints because they can be thought of as the most "extreme" case of credit constraints. Specifically, when agents face collateral constraints, borrowing is tightly constrained by the level of net worth, and so the productivity gap between constrained and unconstrained agents is at its largest. This would in principle generate large amplification effects because when agents differ significantly in their marginal productivity, redistribution of the productive asset can increase output substantially.

The reminder of the paper is organized as follows. Section 2 presents our basic model economy. In Section 3 we characterize the dynamics of the model and derive the conditions under which monotonic saddle-path stability holds for the special case of CRRA utility, and Cobb-Douglas production function. We also present and discuss numerical simulations. We first consider the case in which agents differ only in their discount factors, and then we allow agents to differ also in the EIS and their capital share. Section 4 concludes.

2 The model

2.1 Economic environment

Consider an economy inhabited by two types of agents who differ in their rate of time preference. Agents may also differ in other dimensions such as the degree of risk aversion or the production technologies. There are two goods in this economy: a durable asset (capital, K), and a non-durable commodity (output, C). Agents maximize their expected lifetime utility as given by

$$E\sum_{t=0}^{\infty}\beta_i^t u_i(c_{it}) \text{ for } i = 1,2$$

where $1 > \beta_1 > \beta_2 > 0$, and c_{it} is consumption of agent i at time t. The momentary utility function, u_i, is assumed to satisfy usual properties. We allow for the possibility that u differ across agent's types. There is a continuum of agents of each type with population size $m_i > 0$, $i = \{1, 2\}$. For simplicity, we normalize $m_2 = 1$ and refer to m_1 as m. Following steady state considerations, we often call agents type 1 lenders and agents type 2 borrowers. Except for the unanticipated shock, there is not uncertainty in the model.

Agent i produces using a concave technology, $f_i(k_i)$, where k_i is capital and $\lim_{k\to\infty} f_i'(k) = \infty$. Similar to u, f may also differ across agent's types. Agents face a budget constraint given by

$$c_{it} + q_t(k_{it+1} - k_{it}) + a_{it} = f_i(k_{it}) + p_t a_{it+1}$$

where q is the price of the capital, a_{it+1} is amount (of consumption good) promised to be paid by the borrower (includes principal plus interest rates) at $t+1$, and p_t is the price of one unit of such promise at time t. Agents behave competitively taken prices as given.[2]

We assume that borrowers can disappear without repaying their loans with no other penalty than losing their capital. As a result, loans need to be secured by the value of the capital, i.e.

$$a_{it+1} \leq q_{t+1} k_{it+1}.$$

Capital is available in a fixed aggregate amount, $\overline{K}$. This assumption can be interpreted as either investment taking a long time-to-build, or as the adjustment costs of investment being very high.[3]

It is useful to rewrite the budget constraint in terms of the present value of the net wealth, $w_{it+1} \equiv p_t\left(q_{t+1}k_{it+1} - a_{it+1}\right)$, and the users cost (or down payment) of capital, $s_t \equiv q_t - p_t q_{t+1}$ as follows:

$$c_{it} + s_t k_{it+1} + w_{it+1} = x_{it} \equiv f_i(k_{it}) + w_{it}/p_{t-1}$$

where x_{it} represents total resources available to agent i at the beginning of period t. The advantage of this formulation is that it reduces the individual state vector to one variable, x_{it}, and the collateral

[2] We exclude the possibility of renting capital. Adding this possibility would not change the perfect-foresight equilibrium path but it would affect how the economy responds to an unanticipated shock. In particular, shocks could be less amplified if capital can be rented.

[3] As will become clear below, this assumption helps the model to generate larger amplification. The harder is to accumulate capital the larger is the response of asset prices to unexpected shocks, and the larger the redistribution of resources.

constraint can now be expressed simply as $w_{it+1} \geq 0$.

2.2 Recursive competitive equilibrium

Let variables in capital letters denote the aggregate quantities corresponding to the variables in lowercase. Thus, $K_i = m_i k_i$, $C_i = m_i c_i$, $X_i = m_i x_i$, etc. Let X be the aggregate state of the world. X is usually the vector describing the distribution of x across agents, i.e. $X = \{X_1, X_2\}$. We will be able to reduce this state vector to an scalar, as will be seen below. Assume that capital and bond prices can be expressed as functions of the aggregate state of the world: $q = q(X)$, and $p = p(X)$. Likewise, suppose that X evolves according to the law of motion $X' = G(X)$.

Each period, agents choose their consumption, c_i, stocks of capital, k_i', and wealth, w_i' so as to solve the following dynamic programming problem,

$$V^i(x_i; X) = \max_{c_i, k_i' \geq 0, w_i' \geq 0} \left\{ u_i(x_i - s(X)k_i' - w_i') + \beta_i V^i(f_i(k_i') + w_i'/p(X); X') \right\} \tag{P1}$$

given the law of motion for the aggregate state, $G(X)$. Let $\mu_i(x_i, X)$ be the Lagrange multiplier associated to the collateral constraint. The constraint for capital never binds due to the properties of f. We now proceed to define a competitive equilibrium for this economy:

Definition 1. A competitive equilibrium is a set of prices $s(X)$, $p(X)$, value functions $V^i(x_i; X')$, allocation rules $c_i(x_i, X)$, $k_i(x_i, X)$, $w_i(x_i, X)$, and aggregate law of motion $G(X)$ such that:

1. $V^i(x_i, X)$, $c_i = c_i(x_i, X)$, $k_i' = k_i(x_i, X)$, and $w_i' = w_i(x_i, X)$, solve problem (P1), given $s(X)$, $p(X)$, and the aggregate law of motion $G(X)$.

2. Capital, goods, and asset markets clear: $\sum_{i=1}^{2} m_i k_i(X_i, X) = \overline{K}$, $\sum_{i=1}^{2} m_i c_i(X_i, X) = \sum_{i=1}^{2} m_i f_i(k_i)$, and $\sum_{i=1}^{2} m_i w_i(X_i, X)/p(X) = q(X)\overline{K}$.

3. The aggregate law of motion is consistent with the individual decision rules.

This completes the description of the economy and the equilibrium concept. It is important to stress three features of the model that make it suitable for our purpose. First, the model is a slight modification of a standard representative-agent economy. If borrowing constraints are eliminated (or discount factors are identical) then the economy will collapse into a standard representative-agent economy. The model is thus designed to highlight the role of collateral constraints as the *sole* cause for amplification and persistence effects.

Second, we make no assumptions to keep the interest rate (the inverse of p_t) constant as other papers in the literature do[4]. We can thus study if changes in the interest rate dampen or enhance the asset price effect usually stressed as the key element behind the amplification. Third, the model requires only a small set of parameters on preferences and technologies: the intertemporal elasticity of substitution, factor shares, discount factors, and the mass of credit-constrained agents. We can use evidence about some of these parameters to impose some discipline in the analysis.

Standard arguments can be used to show that the solution to (P1) is characterized by the following optimality conditions:

$$u_i'(c_i)s(X) = \beta_i f_i'(k_i')u_i'(c_i') \tag{1}$$

$$u_i'(c_i)p(X) = \beta_i \left[u_i'(c_i') + \mu_i\right] \tag{2}$$

$$c_i = x_i - s(X)k_i' - w_i' \tag{3}$$

$$\mu_i w_i = 0,\ \mu_i \geq 0,\ \ w_i' \geq 0 \tag{4}$$

The first condition equates the marginal cost of holding capital to its marginal benefit. The second condition states that unconstrained agents equate the marginal benefit of borrowing to its marginal cost. On the other hand, the marginal benefit of borrowing is larger than marginal cost for constrained agents.

Notice that if there were no collateral constraints, equations (1) and (2) imply that production would be efficient, i.e., all agents have the same marginal product of capital. The distribution of

[4]Kiyotaki and Moore (1997) and Kiyotaki (1998) assume linear preferences or technologies, Kocherlakota (2000) assumes a small open economy.

capital in that case satisfies

$$f_1'\left(\left(\overline{K}-K^e\right)/m\right)=f_2'(K^e)$$

where K^e is the capital held by impatient agents. Also, if the production functions are identical for both types of agents, then $K^e=\frac{\overline{K}}{1+m}$, so that all agents would hold the same amount of capital.

2.3 Steady State

Let α_i denote the steady state capital share of output for agents type i. The following proposition summarizes the main properties of the steady state.

Proposition 1. There exist a unique steady state. In steady state impatient agents are credit constrained, and their capital holdings satisfy $K_2^*<K^e$. In addition, the following equations hold:

$$p^*=\beta_1$$

$$\frac{f_2'(K_2^*)}{f_1'\left(\left(\overline{K}-K_2^*\right)/m\right)}=\frac{\beta_1}{\beta_2}>1$$

$$A_2^*=q^*K_2^*=\frac{\beta_1}{1-\beta_1}\frac{\beta_2}{\beta_1}\alpha_2Y_2^*$$

$$s^*=\beta_1f_1'\left(\left(\overline{K}-K_2^*\right)/m\right)$$

$$q^*=\frac{s^*}{1-\beta_1}$$

$$C_1^*=mf_1\left(\left(\overline{K}-K_2^*\right)/m\right)+s^*K_2^*$$

$$C_2^*=f_2(K_2^*)-s^*K_2^*$$

Proof: In equilibrium agents of at least one type are not credit constrained. Therefore, equation (2) evaluated at the steady state implies that

$$p^*\geq\beta_i \text{ for } i=1,2 \text{ and } p^*=\beta_i \text{ for at least some } i.$$

Since $\beta_1 > \beta_2$, it follows that $p^* = \beta_1$ and $p^* > \beta_2$. Thus, $\mu_2^* > 0$, i.e., impatient agents are credit constrained. In addition, equation (1) evaluated at steady state implies that

$$\frac{f_2'(k_2^*)}{f_1'(k_1^*)} = \frac{f_2'(K_2^*)}{f_1'\left(\left(\overline{K} - K_2^*\right)/m\right)} = \frac{\beta_1}{\beta_2} > 1$$

Thus $K_2^* < K^e$. The remaining equations are easy to derive from equations (1) through (3).

The first equation of Proposition 1 states that the steady-state interest rate is completely determined by the discount factor of the patient agents. The second and third equations stress the role played by β_2 in the model: it determines the degree of inefficiency, i.e. the gap in marginal productivities, as well as the debt to output ratio of the constrained agents. A lower β_2 increases the gap in marginal productivities and reduces the debt to output ratio.

Figure 1 illustrates the determination of the steady state of the economy. The efficient allocation with no debt-enforcement problem would imply $K_2^* = K^e$. Notice that in this case, since agents differ in their discount factor, impatient agents will eventually end up with zero consumption. The existence of credit constraints reduces the borrower's capital holdings to $K_2^* < K^e$ and, more importantly, induces a gap in the marginal productivities. This gap is crucial for the model to generate amplification effects. If marginal products were equal in equilibrium, then marginal changes in the distribution of capital would have no effect on output.

3 Dynamics

The previous discussion shows that shocks are amplified as long as some agents are constrained. One possible situation is that along the equilibrium path agents may be constrained only for a while. In order to simplify the analysis and enhance the amplification effects we focus on economies in which constrained agents are always constrained, i.e., $W_{2t}(X_t) = 0$ for all t. [5] The following lemma shows that this is in fact the case if the steady state is globally saddle-path stable and monotonic, and $X_{20} \leq f_2(K^e)$. By monotonic we mean that X_{2t} monotonically converges to X_2^*. Thus, Lemma

[5] In equilibrium, $W_2(X)$ is only function of the aggregate state of the world, X.

1 states that if borrowers start with a level of resources below their first-best level, then they will always remain below that level.

Lemma 1. Suppose the steady state exhibits monotonic global saddle-path stability and $X_{20} \leq f_2(K^e)$. Then, $W_{2t}(X_{2t}) = 0$ for all t.

Proof: From Proposition 1, it follows that $X_2^* < f_2(K^e)$. Since the steady state is monotonic and saddle-path stable, then $X_{2t} < f_2(K^e)$ implies $X_{2t+1} < f_2(K^e)$. Thus $X_{2t} < f_2(K^e)$ for all t. This inequality also implies that $f_2(K_{2t}) < f_2(K^e)$, or $K_{2t} < K^e$. Thus, $f_2'(k_{2t}) > f_1'(k_{1t})$ for all t. Equations (1), (2) yield

$$f_i'(k_{it+1}) = \frac{s(X_t)}{p(X_t) - \beta_i \frac{\mu_{it}}{u_i'(c_{it})}}.$$

The last two conditions imply that $\mu_{1t} = 0$ and $\mu_{2t} > 0$ for all t. Thus, borrowers are credit constrained for all $t > 0$.

Monotonic saddle-path stability also guarantees that the rational expectations equilibrium is unique, and that deviations from the equilibrium are persistent. Uniqueness and persistence are usually properties of the rational expectations equilibrium of frictionless economies, but economies with frictions may also have these properties as is the case of KM or Kocherlakota (2000). Our model, however, may exhibit multiple equilibria and/or cyclical behavior, or may have no equilibria at all. It turns out, however, that for plausible parametrizations our model displays monotonic saddle-path stability as we show below.

From now on we assume that the conditions of Lemma 1 hold so that borrowers are always credit constrained. We do not provide conditions to guarantee monotonic global saddle path stability, but rather derive conditions under which such property holds locally around the steady state. These conditions are provided in the next section. We also comment on the cases in which these conditions do not hold.

Along credit-constrained paths the aggregate state vector can be reduced to only one variable, X_2. In that case $X_2 = f_2(K_2)$ so that K_2 is a sole function of X_2, i.e., $K_2 = K_2(X_2)$. We can then solve for X_1 as an implicit function of X_2 by using the definition $X_1 = mf_1((K - K_2(X_2)/m) + q(X_1, X_2)\overline{K}$. Thus, along credit-constrained paths the aggregate state is described by $X_2 = f_2(K_2)$. For convenience we define the aggregate state as $X \equiv K_2$ rather than $f_2(K_2)$. Let $K_2' \equiv G(X)$ be the aggregate law of motion of K_2.

When borrowers are credit constrained, their budget constraint (3) becomes

$$C_2(X) = f_2(X) - s(X)G(X). \tag{5}$$

In addition, using this equation along with the aggregate resource constraint results in

$$C_1(X) = mf_1\left(\frac{\overline{K} - X}{m}\right) + s(X)G(X). \tag{6}$$

Now, using equation (1) along with (5) and (6), we obtain a two dimensional system of functional equations in $G(X)$ and $s(X)$:

$$u_2'\left[f_2(X) - s(X)G(X)\right]s(X) = \beta_2 f_2'(G(X))u_2'\left[f_2(G(X)) - s(G(X))G(G(X))\right] \tag{7}$$

and

$$\begin{aligned} & u_1'\left(f_1\left(\frac{\overline{K} - X}{m}\right) + \frac{s(X)G(X)}{m}\right)s(X) \\ = \; & \beta_1 f_1'\left(\frac{\overline{K} - G(X)}{m}\right)u_1'\left(f_1\left(\frac{\overline{K} - G(X)}{m}\right) + \frac{s(G(X))G(G(X))}{m}\right). \end{aligned} \tag{8}$$

This system of two functional equations summarizes the equilibrium of the model. It is clear from this system that the functional equations only depend on the capital stock of borrowers, $X = K_2$.[6]

In order to analyze the dynamics of the model, we linearize equations (7) and (8) around the steady state and solve the linear system by the method of undetermined coefficients. Let $\omega_z = \frac{\partial Z}{\partial X}$

[6]This holds as long as the solution to this system is unique, as guaranteed by the conditions of Lemma 1. Otherwise, multiple equilibria may arise.

be the response of a control variable $Z = \{C_1, C_2, q, p, s\}$ to changes in the state variable, X; and let $\omega_x = \frac{\partial G}{\partial X}$ describe the motion of the state variable. Define $u_i \equiv u_i(c_i)$ and $f_i \equiv f_i(k_i)$. Linearizing the two equations around the steady state, using the fact that $s^* = \beta_2 f_2' = \beta_1 f_1'$, and collecting terms we obtain

$$s\omega_x^2 + x\omega_s\omega_x = \epsilon_2\omega_x + \zeta_2\omega_s - f_2' \tag{9}$$

$$s\omega_x^2 + x\omega_s\omega_x = \epsilon_1\omega_x + \zeta_1\omega_s - f_1'$$

where $\epsilon_i = \left(\frac{f_i'' u_i'}{f_i' u_i''} + f_i' + s\right) > 0$, and $\zeta_i = \left(x - (-1)^i \frac{m_i u_i'}{s u_i''}\right) \gtreqless 0$. Equating these two equations and solving for ω_s we obtain

$$\omega_s = \frac{(\epsilon_2 - \epsilon_1)}{\zeta_1 - \zeta_2}\omega_x + \frac{f_1' - f_2'}{\zeta_1 - \zeta_2}. \tag{10}$$

This equation provides the solution for ω_s once ω_x is determined. Substituting this result into (9) we obtain a second order polynomial equation which roots provide the solutions for ω_x

$$\pi(\omega_x) \equiv \theta_1\omega_x^2 + \theta_2\omega_x + \theta_3 = 0 \tag{11}$$

where $\theta_1 = s(\zeta_1 - \zeta_2) + x(\epsilon_2 - \epsilon_1)$; $\theta_2 = \epsilon_1\zeta_2 - \zeta_1\epsilon_2 + x(f_1' - f_2')$ and $\theta_3 = \zeta_1 f_2' - \zeta_2 f_1'$. Provided the solutions for ω_x, and ω_s we can solve for ω_{c1}, ω_{c2}, ω_q, and ω_p, as shown in the Appendix.

3.1 Symmetric case

To derive more precise results it is convenient to assume specific functional forms at this point. Suppose that $f(k) = k^\alpha$ and $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, so that $1/\sigma$ is the intertemporal elasticity of substitution. Note that we assume that α and σ are the same for both agents, so that they only differ in their discount factors. We call this case "symmetric". Next section we study the case in which α and σ differ across agents.

The following proposition characterizes the restrictions on the parameters $(\sigma, \alpha, \beta_1, \beta_2, m)$ such that the steady state exhibits monotonic saddle-path stability.

Proposition 2. The steady state exhibits monotonic saddle-path stability if and only if

$$\sigma < \widehat{\sigma}\left(\alpha, m, \beta_1, \beta_2\right) \equiv 1 + \frac{1 + m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1-\alpha}}}{\alpha\left(\beta_1 - \beta_2\right)} \tag{12}$$

and

$$\begin{aligned} \sigma \quad < \quad & \widetilde{\sigma}\left(\alpha, m, \beta_1, \beta_2\right) \equiv \frac{\beta_1}{\beta_1 - \beta_2}\frac{1}{\alpha}\frac{1}{m} \\ & \left[m^2\left(\frac{\beta_1}{\beta_2}\right)^{\frac{\alpha}{1-\alpha}} + m\left((1-\alpha)\frac{\beta_2}{\beta_1} + \alpha + \alpha\left(1-\alpha\right)\beta_2\right) + \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{1-\alpha}}(1-\alpha)\alpha\beta_2\right] \end{aligned} \tag{13}$$

Proof: See Appendix.

These conditions state that very large values of σ (i.e., close to zero EIS), low values of m, and certain values of α are not admissible. In other words, the model may display multiple equilibria, instability, and/or cycles in those cases. It is easy to derive the following properties of $\widehat{\sigma}\left(\alpha, m, \beta_1, \beta_2\right)$ and $\widetilde{\sigma}\left(\alpha, m, \beta_1, \beta_2\right)$:

Lemma 2. Properties of $\widehat{\sigma}\left(\alpha, m, \beta_1, \beta_2\right)$ and $\widetilde{\sigma}\left(\alpha, m, \beta_1, \beta_2\right)$:

- $\widehat{\sigma}\left(\alpha, m, \beta_1, \beta_2\right) > 1$ and $\widetilde{\sigma}\left(\alpha, m, \beta_1, \beta_2\right) > 1$
- $\widehat{\sigma}\left(0, m, \beta_1, \beta_2\right) = \widehat{\sigma}\left(1, m, \beta_1, \beta_2\right) = \infty$
- $\widetilde{\sigma}\left(0, m, \beta_1, \beta_2\right) = \widetilde{\sigma}\left(1, m, \beta_1, \beta_2\right) = \infty$
- $\widehat{\sigma}\left(\alpha, m, \beta_1, \beta_1\right) = \widetilde{\sigma}\left(\alpha, m, \beta_1, \beta_1\right) = \infty$
- $\frac{\partial \widehat{\sigma}(\alpha, m, \beta_1, \beta_1)}{\partial m} > 0$ and $\frac{\partial \widetilde{\sigma}(\alpha, m, \beta_1, \beta_1)}{\partial m}$ is not monotonic in m.

Special economies that satisfy these conditions include an economy with log utility function ($\sigma = 1$); a representative-agent economy ($\beta_1 = \beta_2$); an economy with a large mass of unconstrained agents ($m \to \infty$); and AK or AL type economies (i.e. $\alpha \simeq 1$ or $\alpha \simeq 0$). On the other hand, the larger the steady-state the productivity gap, $\beta_1 - \beta_2$, the smaller the set of parameters $(\alpha, m, \beta_1, \beta_2)$ that is consistent with monotonic saddle-path stability.

We now study the response of the model economy to an unanticipated shock by using mostly numerical simulations. Assume that the economy is at the steady state at time zero, and that a one-time unexpected productivity shock occurs so that production is z_0 times the steady-state production. To find the equilibrium path we assume that the recursive solution holds from time one onward, and find the solution at time zero backwards by using the recursive solution for time $t = 1$. Details are discussed in the appendix.

The productivity shock z_0 provides more resources to both constrained and unconstrained agents. Since this is a temporary shock, both types of agents save part of the extra resources in order to smooth consumption. The difference between the two types of agents is that the unconstrained are indifferent between buying capital or bonds because they have an interior solution, while the constrained, who are borrowers, will smooth the shock by buying capital. In fact, constrained agents are in a corner solution so that the only way to borrow more is to buy more capital. Since borrowers' marginal product of capital is higher, aggregate output increases following the productivity shock. Thus, the fundamental channel behind amplification is the redistribution of capital toward agents with high productivity.

The two main variables of interest are amplification and persistence. We define amplification as the elasticity of output in period one with respect to a productivity shock in period zero, ϵ_{YZ}.[7] Persistence is measured by ω_x.

3.1.1 Amplification

Output in period one can only vary if the state variable X_1 varies. We can then write ϵ_{YZ} as the product of two components: the elasticity of output at time one with respect to X_1, ϵ_{YX} , times

[7] The elasticity of output in period zero with respect to a productivity shock in period zero is always 1 in this model.

the elasticity of X_1 with respect to Z_0, ϵ_{XZ}

$$\epsilon_{YZ} = \epsilon_{YX}\epsilon_{XZ}. \tag{14}$$

ϵ_{XZ} represents the redistribution of capital toward constrained agents. Using the definition of total product, it follows that

$$\begin{aligned} \epsilon_{YX} &= \left(f_2' - f_1'\right)\frac{X}{Y} \\ &= \underbrace{\frac{(f_2' - f_1')}{f_2'}}_{\text{productivity gap}} \cdot \underbrace{\alpha}_{\text{collateral share}} \cdot \underbrace{\frac{Y_2}{Y}}_{\text{output share}} \end{aligned}$$

This equation suggests that ϵ_{YX} is typically a small number. For example, if constrained agents were 50% more productive, produce half of the total output, and have a capital share of 1/2, then $\epsilon_{YX} = \frac{1}{8}$. One can in principle try to increase ϵ_{YX} by inducing a larger productivity gap and a larger output share, given certain plausible value for the capital share. There is a limit, however, to how much can be accomplished this way due to the trade-off between the productivity gap and the output share. Under standard concave technologies, a large productivity gap requires borrowers to hold little capital. But if borrowers hold little capital, then $\frac{Y_2}{Y}$ is small. Thus, if the model is to produce significant amplification, then ϵ_{XZ} must be significantly large to compensate for the small value of ϵ_{YX}. In other words, significant amplification requires a very large redistribution of capital toward constrained agents. However, a large redistribution of capital toward constrained agents is not sufficient to guarantee significant output amplification.

KM show that in their model ϵ_{XZ} is significantly large, in the order of $\frac{1}{1-\beta_1}$. They, however, do not discuss at all the size of ϵ_{YZ} or ϵ_{YX} in their model. Their claims about the power of their propagation mechanism refer only to the redistributive properties of their model but not to its ability to generate large responses *in output*. It turns out, however, than under certain parameterization ϵ_{YX} can be made arbitrarily close to 1 in KM, which implies a large elasticity of output to the shock. The reason is that constrained agents in their model use a linear technology which avoids the trade-off between the productivity gap and the output share.[8]

Figure 2 illustrates both the magnitude of output amplification and the size of capital redistribution in our model for different pairs $(\alpha, 1/\sigma)$, and for $\beta_1 = 0.99$, $\beta_2 = 0.9\beta_1$, and $m = 0.5$. As shown below, our main results are not sensitive to the particular choice of parameters. There are at least four important observations from this figure: *(i)* Output amplification is "small" (below one) for most parameter configurations. *(ii)* There are configurations of parameters that produce significant amplification (larger that one). They require a low EIS and large capital share. *(iii)* The transition between the area of low to high amplification is sharp: amplification is generally small, but it quickly changes to be very large for certain configurations of parameters. *(iv)* Although capital redistribution is also "small" for a large set of parameters, it responds more than output and can be quite sizeable when the EIS is low and α is large.

An additional important observation is obtained looking more closely into the area of largest output amplification, around the hill of Figure 2. Figure 3.a. shows a top perspective of this area. The white hump-shaped area corresponds to $(\alpha, 1/\sigma)$ parameters for which monotonic saddle-path stability does not hold. In other words, parameters on this hump violate the conditions stated in Proposition 2. Notice how the largest amplification, which corresponds to the darkest shade, is right at the border of the hump. Thus, the configurations of parameters that produce the largest amplification are at the edge of the space of monotonic saddle-path stability.

Figure 3.b. illustrates the types of dynamic behavior generated by the parameters on the hump-shaped area. First, the top-left part of the hump corresponds to the area in which the only stable root is negative. Recall that in Proposition 2 we have ruled out these roots to avoid non-monotonic dynamics. This area is not interesting because it implies jagged dynamics, which are clearly non-plausible. Second, notice that most of the hump corresponds to unstable roots, i.e., an area where there is no forward looking equilibrium. Finally, there are two stretches that correspond to multiple equilibria cases, i.e. two positive stable roots, and two complex roots. Even though the cases of multiple equilibria exhibit stable dynamics, one can easily eliminate these cases by allowing agents

[8] In addition to the linear technology, a low saving rate is required to generate large amplification in KM.

to differ not only in β but also in α and σ, as we do next section.

The previous observations cast doubts on the ability of collateral constraints to produce significant amplification for two main reasons. First, large amplification is not a robust result of the model. The model produces large amplification only as a "knife-edge" type of result: it requires a very particular combination of parameters at the edge of the space of monotonic saddle-path stability. In other words, a small change in parameters can either reduce the amplification dramatically, produce jagged dynamics, instability, or multiple equilibria.

Second, the parameters required to generate large amplification are not empirically plausible. On the one hand, the share of collateral in the production function is probably lower than 1/3 which is approximately the capital share of output in the U.S. But the results in Figure 2 (and Figure 5 below) indicate that the capital share must be at least 0.5 in order to obtain some significant amplification. In addition, the EIS in the U.S. is probably well above 0.3, as recently documented by Vissing-Jorgensen (2002). However, the results in Figure 2 (and Figure 5 below) indicate that large amplification requires the EIS to be well below 0.2.

It is easy to understand why large amplification requires a large capital share. It directly affects the elasticity of output to the shocks (see equation (14)). It is less obvious to explain why a lower EIS increases the amplification. To understand this, notice first that the shock provides more resources to all agents in the economy. Since the shock is temporary, all agents save part of the extra resources in order to smooth consumption. Unconstrained agents are indifferent between buying capital or bonds because they have an interior solution. However, constrained will smooth consumption by buying capital. They are in a corner solution so that the only way to borrow more is to buy more capital. Since borrowers' marginal product of capital is higher, aggregate output increases following the productivity shock. Consider now the effect of lowering the EIS. In that case constrained agents spend a larger fraction of the unexpected resources buying capital because the smoothing motive becomes stronger. Thus, a lower EIS implies an even larger redistribution of capital toward the more productive agents, and a larger amplification.

Up to now we have illustrated the magnitude of amplification for pairs $(\alpha, 1/\sigma)$ but for a given mass of unconstrained agents, m, and productivity ratio, β_2/β_1. How does amplification depend on m and β_2/β_1? Figure 4 illustrates this relationship for given values of α, σ and β_1. Notice that amplification is non-monotonic in β_2/β_1: it first increases and then decreases. If β_2 is very low, then borrowers own very little capital in the economy and their effect on aggregate variables is small. Therefore, amplification effects are low. As β_2 increases, borrowers own a larger fraction of capital in the economy, and so amplification effects become more important. However, as β_2 gets closer to β_1 then the productivity differentials start to vanish, so that the amplification is small. The impact of m on the amplification is mixed but overall a small m seems to help amplification. However, m cannot be arbitrarily small because when there are too many credit-constrained agents and the productivity shock occurs, there will be a large boom in demand for credit, and the interest rate may increase so much that the conditions for saddle-path stability may be violated.

It may seem important at this point to come up with some empirically plausible values for β_2/β_1 and m. However, it is hard to find convincing information about this parameters. Fortunately, we do not really need to know much about these parameters for our purposes. We can choose β_2/β_1 and m to maximize the amplification (ϵ_{YZ}) for each pair $(\alpha, 1/\sigma)$ given a plausible value for β_1. This procedure provides an upper bound for ϵ_{YZ}. If the upper bound is small, then we must conclude the model cannot generate much amplification. Figure 5 depicts the outcome of this exercise given $\beta_1 = 0.99$. It confirms that for empirically plausible values of α and σ the amplification is almost nil. Large amplification requires a very large α and a very low EIS.

3.1.2 Prices, persistence, and other variables

In this section we discuss the behavior of some key variables of the model to gain further insight into the origin of the amplification effects. Figure 6 presents the impulse responses of the borrowers' output Y, capital stock K_2, bond prices p, capital prices q, the users cost of capital s, and the split of Y into C_1 and C_2. All values are percentage deviations from the steady state. The parameters used for this simulation are $\beta_1 = 0.99$, $\beta_2 = 0.9\beta_1$, $\alpha = 0.8$, $\sigma = 15$ and $m = 0.3$. First notice that at the time of the shock $t = 0$, Y increases by 1%, which is the magnitude of the shock, while next period $t = 1$, output reaches a maximum amplification of about 1.2%. This 'large' amplification is obtained using relatively high values for α and σ.

The panel for K_2 clearly shows that the large redistribution of capital across agents is key to generate amplification. Borrowers increase their capital holdings by about 30%. Part of this increase is explained by increase in the value of the collateral, which increases around 30% the period after the shock. This large price increase could have produce a much larger redistribution of capital but the large increase of the interest rate, of around 20%, partially offsets the price effect.

There is an interesting split of Y into C_1 and C_2: in the period of the shock, borrowers are both consuming more and buying more capital. In fact, C_2 increases around 1%, almost the full increase in Y. Instead, lenders increase consumption very little in the period of the shock, but they wait until next period to enjoy the higher returns in bonds. In effect, C_1 barely increases at $t = 0$, but it is around 0.6% higher than the steady state in $t = 1$. In summary, as in KM, most of the action in this model occurs in the period of the shock and is associated to a large redistribution of capital from lenders to borrowers. This redistribution is so large that prices react substantially.

Finally, it turns out that persistence in the model is generally small, and it is increasing in α. The reason is that the effects of the shock are persistent in this model as long as borrowers' net worth is high enough to allow them to continue buying capital. The larger the α, the more extra output borrowers obtain from an extra unit of capital, and the higher their net worth is. It is interesting to note that the region of parameters for which amplification is largest corresponds to close-to-zero persistence in the model. This is so because the largest amplification is achieved with a substantial redistribution of capital toward borrowers, which implies a large increase in the interest rate that makes this amplification effects short lived.

3.2 Asymmetric Case

Up to this point we have discussed simulations in which agents only differ in their discount factors. One of the conclusions from these simulations is that large amplification can be obtained with a low EIS, and a large, but not too-close-to-one capital share. The evidence on the value of σ is controversial, and some of it indicates that the EIS is very close to zero, at least for a set of agents in the economy (Guvenen, 2002). Thus, an interesting exercise would be one in which we allow agents to differ in σ. In particular, in order to "help the model" generate large amplification, we would like borrowers to have a low EIS.

Another interesting simulation is to allow for different α's across borrowers and lenders. It is generally assumed that for the U.S. the aggregate capital share is $\alpha = 0.3$, but as reported by Barro and Sala-i-Martin (1995), α can be in a range from 0.45 to 0.69 for developing countries.[9] If we want to "help the model" generate large amplification, then we can let borrowers have a high α, and lenders a low α, so that the aggregate capital share is consistent with the empirical evidence.

Figure 7 shows the amplification achieved when agents differ in β, σ and α. In particular, $\alpha_1 = 0.3$, $\sigma_1 = 0.1$, $m = 0.5$, $\beta_1 = 0.99$, and $\beta_2 = 0.9\beta_1$. This figure confirms our previous finding that amplification is typically small. Large amplification requires a very large α_2 and a very low EIS for the credit-constrained agent. Finally, notice that all parameter combinations in Figure 7 guarantee monotonic saddle-path stability, i.e. the hump-shaped area of Figure 3.a. has disappeared.

4 Concluding comments

The purpose of this paper is to evaluate the role of collateral constraints as an amplification mechanism of exogenous shocks to the economy. In particular, we analyze a simple deterministic economy that incorporates the main mechanism proposed by KM. According to this mechanism, what causes amplification is the fact that a group of agents in the economy are credit-constrained and have a higher marginal product of capital. Thus, adverse shocks to the net worth of constrained agents negatively affect investment in collateral, output and asset prices. The fall in the value of the collateral worsens the downturn because it further limits the ability of constrained agents to borrow.

We analyze how amplification changes for different parameters when we allow for standard utility and production functions. Our approach is to "help the model" generate amplification by

[9] See Table 10.8, page 380-1.

analyzing equilibrium paths along which a group of agents is always against the constraint. Further, we also "help" it by considering only unexpected shocks, ruling out a market for renting capital, and preventing capital from being accumulated. The idea is that if even under these "favorable" conditions the model does not generate amplification, then it would be difficult for more general, less-stylized models with collateral constraints to do so.

As the simulations indicate, in this deterministic model large amplification can be obtained only with the "right" combination of a low EIS; a large, but not too-close-to-one capital share; and a sizeable, but not too-close-to-one share of constrained agents. Thus, unless one has this right combination of parameters, collateral constraints can generate amplification when compared with perfect-market models, but this amplification is small.

Our findings would still hold if agents were heterogenous in other dimensions. Here we introduce heterogeneity in the discount factors, but this is nonessential. Any heterogeneity that induces differences in productivity across agents would produce similar results. This is so because the fundamental channel to produce amplification is the redistribution of a productive asset from lower to higher-productivity agents. In general, when technology exhibits marginal decreasing returns in the productive asset, the largest output amplification would be attained when this asset is transferred to agents who hold a very small fraction of it. However, by the same token, since high-productivity agents hold a very small fraction of the productive asset, their impact on aggregate production is small. All in all, our results show that collateral constraints *by themselves* are not enough to account for the large fluctuations of output observed in the data.

References

[1] Barro, R. and X. Sala-i-Martin [1995]: Economic Growth. McGraw-Hill.

[2] Bernanke, B. and M. Gertler [1989]: "Agency Costs, Net Worth, and Business Fluctuations," in *American Economic Review*, 79, March, 14-31.

[3] Bernanke, B., M. Gertler and S. Gilchrist [1999]: "The Financial Accelarator in a Quantitative Business Cycle Framework," in *Handbook of Macroeconomics*, Vol. 1, edited by J.B. Taylor and M. Woodford, 1341-1393.

[4] Caballero, R. and A. Krishnamurthy [2001]: "International and Domestic Collateral Constraints in a Model of Emerging Market Crises," in *Journal of Monetary Economics*, 48, 3, December.

[5] Carlstrom, C. and T. Fuerst [2000]: "Monetary Shocks, Agency Costs and Business Cycles". *Carnegie-Rochester Public Policy Conference*, April 14-15.

[6] Carlstrom, C. and T. Fuerst [1997]: "Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis," in *American Economic Review*, 87, December, 893-910.

[7] Cochrane, J. [1994]: "Shocks," in *Carnegie-Rochester Conference Series on Public Policy*, 41, December.

[8] Cooley, T., R. Marimon, and V. Quadrini [2001]: "Aggregate Consequences of Limited Contract Enforceability," Mimeo.

[9] Guvenen, Fatih [2002]: "Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective," Rochester Center for Economic Research, Working Paper # 491, May.

[10] Kiyotaki, N. and J. Moore [1997]: "Credit Cycles," in *Journal of Political Economy*, 105, No.2, 211-248.

[11] Kiyotaki, N.[1998]: "Credit and Business Cycles," in *The Japanese Economic Review*, 49, No. 1, March.

[12] Kocherlakota, N. [2000]: "Creating Business Cycles Through Credit Constraints," in *Federal Reserve Bank of Minneapolis Quarterly Review*, 24, No. 3, Summer.

[13] Krishnamurthy, A. [1998]: "Collateral Constraints and the Amplification Mechanism," Mimeo.

[14] Paasche, B. [2001]: "Credit Constraints and International Financial Crises," in *Journal of Monetary Economics*, 48, 3, December.

[15] Summers, L. [1986]: "Some skeptical observtions on real business cylce theory," in *Federal Reserve Bank of Minneapolis Quarterly Review*, 10, Fall.

[16] Vissing-Jorgensen, A. [2002]: "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution," in *Journal of Political Economy*, 110, 4.

A Linearization

Provided the solutions for ω_x, and ω_s from equations (9) and (10), the remaining variables can be solved for. First, ω_{c_1} and ω_{c2} can be found from equations (5) and (6)

$$\omega_{c_1} = -f_1' + \beta_1 f_1' \omega_x + x\omega_s = x\omega_s - f_1' (1 - \beta_1 \omega_x) \tag{15}$$

$$\omega_{c_2} = f_2' - \beta_2 f_2' \omega_x - x\omega_s = -x\omega_s + f_2' (1 - \beta_2 \omega_x) \tag{16}$$

Note that $\omega_{c_1} + \omega_{c_2} = f_2' - f_1'$. To find ω_q, define $W(X) \equiv u' \left(\frac{C_1(X)}{m}\right) q(X)$. From (1) and the definition of $s(X)$ we obtain

$$W(X) = \beta_1 u' \left(\frac{C_1(G(X))}{m}\right) f' \left(\frac{\overline{K} - G(X)}{m}\right) + \beta_1 W(G(X))$$

Linearize this expression around the steady state and collect terms to obtain

$$\omega_w = \frac{\beta_1 \omega_x}{1 - \beta_1 \omega_x} \frac{1}{m} \left[u_1'' f_1' \omega_{c_1} - u_1' f_1''\right]. \tag{17}$$

Next, use the definition of $W(X)$ to solve for ω_q as

$$\omega_q = \frac{1}{u_1'} \left(\omega_w - \frac{\beta_1 f_1'}{1 - \beta_1} \frac{1}{m} u_1'' \omega_{c_1}\right). \tag{18}$$

Finally, the solution for ω_p reads

$$\omega_p = -\beta_1 \frac{u_1''}{u_1'} \frac{1}{m} \omega_{c_1} (1 - \omega_x). \tag{19}$$

B An unanticipated shock

Assume that the economy is in steady state at time zero. At that moment, a one-time unanticipated productivity shock occurs so that total production is z_0 times the steady state production. To find the equilibrium path notice that the recursive solution holds true from time $t = 1$ on. Given the solution at time $t = 1$, time zero can be solved backwards. The equations that describe the equilibrium at time zero are (7), (8), (2), and the definition of s_t. Let X_0 and X_1 be the state variable at time 0 and time 1 respectively. Then the following equations describe the solution at time zero

$$u_2' \left[z_0 f_2(X_0) + (q_0 - q^*) X_0 - X_1 s_0\right] s_0 = \beta_2 f_2'(X_1) u_2' \left[C_2(X_1)\right]$$

$$u_1' \left[z_0 f_1(\frac{\overline{K} - X_0}{m}) + \frac{X_1 s_0 - (q_0 - q^*) X_0}{m}\right] s_0 = \beta_1 f_1' \left(\frac{\overline{K} - X_1}{m}\right) u_1' \left[\frac{C_1(X_1)}{m}\right]$$

$$u_1' \left[z_0 f_1(\frac{\overline{K} - X_0}{m}) + \frac{X_1 s_0 - (q_0 - q^*) X_0}{m}\right] q_0 = \beta_1 u_1' \left(\frac{C_1(X_1)}{m}\right) \left[f_1' \left(\frac{\overline{K} - X_1}{m}\right) + q(X_1)\right].$$

This is a system of three equations in three unknowns: X_1, s_0, and q_0. Define $\widehat{x}_t = K_{2t} - K_2^*$,

$\widehat{s}_t = s_t - s^*$, $\widehat{q}_t = q_t - q^*$, and $\widehat{z}_t = z_t - 1$. The following is the linearized version of the previous system, where we used the facts that $q^* = \beta_1 [f_1' + q^*]$ and $\omega_{c_1} + \omega_{c_2} = f_2' - f_1'$

$$\left(\epsilon_2 + \omega_{c_2} - f_2'\right) \widehat{x}_1 - x\widehat{q}_0 + \zeta_2 \widehat{s}_0 = f_2 \widehat{z}_0 \tag{20}$$

$$\left(\epsilon_1 + \omega_{c_2} - f_2'\right) \widehat{x}_1 - x\widehat{q}_0 + \zeta_1 \widehat{s}_0 = -m f_1 \widehat{z}_0$$

$$\left((1-\beta_1) \frac{u_1'}{f_1' u_1''} \left[f_1'' - m\omega_q\right] - \omega_{c_1} + s\right) \widehat{x}_1 + \left(\frac{m u_1'}{q u_1''} - x\right) \widehat{q}_0 + x\widehat{s}_0 = -m f_1 \widehat{z}_0 \tag{21}$$

The two first equations can be use to find a solution for $\widehat{s}_0$

$$\widehat{s}_0 = \frac{\epsilon_2 - \epsilon_1}{\zeta_1 - \zeta_2} \widehat{x}_1 - \frac{m f_1 + f_2}{\zeta_1 - \zeta_2} \widehat{z}_0 \tag{22}$$

Equations (20), (21), and (22) can be used to solve time zero values. One can use these solutions to find $\widehat{c}_{10}$, $\widehat{c}_{20}$ and $\widehat{p}_0$ as follows

$$\widehat{c}_{10} = m f_1 \widehat{z}_0 - x\widehat{q}_0 + s\widehat{x}_1 + x\widehat{s}_0$$

$$\widehat{c}_{20} = f_2 \widehat{z}_0 + x\widehat{q}_0 - s\widehat{x}_1 - x\widehat{s}_0$$

$$\widehat{p}_0 = \beta_1 \frac{u_1''}{u_1'} \frac{1}{m} \left(\omega_{c_1} \widehat{x}_1 - \widehat{c}_{10}\right).$$

C Proof of Proposition 2

We want to derive restrictions on the parameters so that the roots of the following polynomial equation

$$\pi(\omega_x) \equiv \theta_1 \omega_x^2 + \theta_2 \omega_x + \theta_3 = 0$$

are positive and guarantee that the steady state exhibits monotonic saddle-path stability. Recall that monotonic saddle-path stability requires that the roots are real, and only one of them less than one.

The idea of the proof is as follows. First we show that it is always the case that $\pi(1) = \theta_1 + \theta_2 + \theta_3 > 0$. Second, we derive conditions under which $\theta_3 < 0$. Note that $\pi(0) = \theta_3$. We need $\theta_3 < 0$ because given that $\pi(1) > 0$, if it was the case that $\theta_3 > 0$ then we would either have multiple equilibria (two real or two complex roots, both stable), or unstable roots. Finally, we derive conditions under which $\theta_1 < 0$. We need $\theta_1 < 0$ because otherwise there would be a negative stable root.

Thus, we derive conditions under which $\pi(\omega_x)$ is initially increasing, crosses the ω_x axis before 1, continues increasing, and then eventually starts decreasing and crosses the ω_x axis for a second time after 1, and tends to minus infinity.

The following are the definitions

$$\theta_1 = s(\zeta_1 - \zeta_2) + x(\epsilon_2 - \epsilon_1)$$

$$\theta_2 = \epsilon_1 \zeta_2 - \zeta_1 \epsilon_2 + x\left(f_1' - f_2'\right)$$

$$\theta_3 = \zeta_1 f_2' - \zeta_2 f_1'$$

$$\epsilon_i = \frac{1-\alpha}{\sigma}\frac{c_i^*}{k_i} + s^* + f_i' = h_i + s^* + f_i'$$

$$\zeta_1 = x - \frac{mc_1^*}{\sigma s^*} \text{ and } \zeta_2 = x + \frac{c_2^*}{\sigma s^*}$$

$$\omega_x = \frac{-\theta_2 \pm \sqrt{\theta_2^2 - 4\theta_1\theta_3}}{2\theta_1}$$

Notice that we need $\theta_2^2 - 4\theta_1\theta_3 \geq 0$ for the roots to be real. The following are some useful results used below

$$\frac{K_1^*}{K_2^*} = m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1-\alpha}}$$

$$\frac{Y_1^*}{Y_2^*} = \frac{\beta_2}{\beta_1}\frac{K_1^*}{K_2^*} = m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{\alpha}{1-\alpha}}$$

$$Y^* = \left(1 + \frac{\beta_2}{\beta_1}\frac{K_1^*}{K_2^*}\right)Y_2^* = \left(1 + m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{\alpha}{1-\alpha}}\right)Y_2^*$$

$$C_2^* = (1-\alpha\beta_2)\,Y_2^*$$

$$C_1^* = Y_1^* + sK_2^* = \left[m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{\alpha}{1-\alpha}} + \alpha\beta_2\right]Y_2^*$$

$$\frac{K_2^*}{Y_2^*}\frac{Y_1^*}{K_1^*} = \frac{\beta_2}{\beta_1}$$

C.1 Solution for $\theta_1 + \theta_2 + \theta_3$

Using the definitions of θ_1, $\theta_2$and θ_3

$$\begin{aligned}
\theta_1 + \theta_2 + \theta_3 &= s(\zeta_1 - \zeta_2) + x(\epsilon_2 - \epsilon_1) + \epsilon_1\zeta_2 - \zeta_1\epsilon_2 + x\left(f_1' - f_2'\right) + \zeta_1 f_2' - \zeta_2 f_1' \\
&= x(h_2 - h_1) + h_1\zeta_2 - h_2\zeta_1 \\
&= h_1\frac{C_2^*}{\sigma s^*} + h_2\frac{C_1^*}{\sigma s^*} = \frac{1-\alpha}{\sigma^2}\left(\frac{C_1^*}{s^* K_1^*}C_2^* + \frac{C_2^*}{s^* K_2^*}C_1^*\right) > 0
\end{aligned}$$

and thus

$$\begin{aligned}
\theta_1 + \theta_2 + \theta_3 &= x(h_2 - h_1) + h_1\left(x + \frac{C_2^*}{\sigma s^*}\right) - h_2\left(x - \frac{C_1^*}{\sigma s^*}\right) \\
&= h_1\frac{C_2^*}{\sigma s^*} + h_2\frac{C_1^*}{\sigma s^*}
\end{aligned}$$

which implies

$$\theta_1 + \theta_2 + \theta_3 = \frac{1-\alpha}{\sigma^2} \left(\frac{C_1^*}{s^* K_1^*} C_2^* + \frac{C_2^*}{s^* K_2^*} C_1^* \right) > 0$$

as desired.

C.2 Solution for θ_3

Using the definition of θ_3

$$\begin{aligned} \theta_3 &= \zeta_1 f_2' - \zeta_2 f_1' = \frac{f_2'}{\beta_1} (\zeta_1 \beta_1 - \zeta_2 \beta_2) \\ &= \frac{f_2'}{\sigma s^* \beta_1} (\sigma s^* K_2^* (\beta_1 - \beta_2) - \beta_1 C_1^* - \beta_2 C_2^*) \end{aligned}$$

From this equation, we conclude that $\theta_3 < 0$ iff

$$\sigma < \widehat{\sigma} \equiv \frac{1}{(\beta_1 - \beta_2)} \frac{\beta_1 C_1^* + \beta_2 C_2^*}{s^* K_2^*}$$

where the right-hand-side does not depend on σ because the steady state does not depend on σ. Solving for $\widehat{\sigma}$

$$\begin{aligned} \widehat{\sigma} &= \frac{1}{(\beta_1 - \beta_2)} \frac{\beta_1 (Y_1^* + Y_2^* - C_2^*) + \beta_2 C_2^*}{s^* K_2^*} \\ &= \frac{1}{(\beta_1 - \beta_2)} \frac{\beta_1 + (\beta_2 - \beta_1)(1 - \alpha\beta_2) + \beta_1 Y_1^* / Y_2^*}{\alpha \beta_2} \end{aligned}$$

or

$$\widehat{\sigma} = \frac{1}{\alpha(\beta_1 - \beta_2)} \left[\frac{\beta_1 + (\beta_2 - \beta_1)(1 - \alpha\beta_2)}{\beta_2} + m \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1-\alpha}} \right]$$

which can be written as

$$\widehat{\sigma}(\alpha, m, \beta_1, \beta_2) = 1 + \frac{1 + m \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1-\alpha}}}{\alpha(\beta_1 - \beta_2)}$$

as shown in the text. Function $\widehat{\sigma}(\alpha, m, \beta_1, \beta_2)$ has the following properties:

- $\widehat{\sigma}(\alpha, m, \beta_1, \beta_2) > 1$
- $\widehat{\sigma}(0, m, \beta_1, \beta_2) = \widehat{\sigma}(1, m, \beta_1, \beta_2) = \infty$
- $\widehat{\sigma}(\alpha, m, \beta_1, \beta_1) = \infty$
- $\frac{\partial \widehat{\sigma}(\alpha, m, \beta_1, \beta_1)}{\partial m} > 0$

Notice that in the representative-agent model, where $\beta_1 = \beta_2$, then $\theta_3 < 0$ always holds regardless of sigma. Notice also that the larger the steady-state productivity gap, i.e. the larger the difference between β_1 and β_2, very low values of σ would not be admissible.

C.3 Solution for θ_1

Using the definition of θ_1

$$\begin{aligned}
\theta_1 &= s\left(x - \frac{C_1^*}{\sigma s^*} - x - \frac{C_2^*}{\sigma s^*}\right) + x\left(\frac{1-\alpha}{\sigma}\frac{C_2^*}{k_2} + s^* + f_2' - \frac{1-\alpha}{\sigma}\frac{C_1^*}{k_1} - s^* - f_1'\right) \\
&= sx\left[-\frac{C_1^* + C_2^*}{\sigma s^* K_2^*} - \frac{1-\alpha}{\sigma}\left(\frac{C_1^*}{sK_1} - \frac{C_2^*}{sK_2^*}\right) + \frac{1}{\beta_2} - \frac{1}{\beta_1}\right]
\end{aligned}$$

From this equation, it follows that $\theta_1 < 0$ iff

$$\frac{1}{\beta_2} - \frac{1}{\beta_1} < \frac{C_1^* + C_2^*}{\sigma s^* K_2^*} + \frac{1-\alpha}{\sigma}\left[\frac{C_1^*}{sK_1} - \frac{C_2^*}{sK_2^*}\right]$$

Denote $\kappa \equiv \frac{K_2^*}{K_1^*} = \frac{1}{m}\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{1-\alpha}}$. Then the previous inequality becomes

$$\begin{aligned}
\sigma &< \widetilde{\sigma} \equiv \frac{\beta_1\beta_2}{\beta_1 - \beta_2}\frac{1}{s^* K_2^*}\left[C_1^* + C_2^* + (1-\alpha)\left(\kappa C_1^* - C_2^*\right)\right] \\
&= \frac{\beta_1\beta_2}{\beta_1 - \beta_2}\frac{1}{s^* K_2^*}\left[Y^* + (1-\alpha)\left(\kappa Y^* - (1+\kappa)\, C_2^*\right)\right] \\
&= \frac{\beta_1}{\beta_1 - \beta_2}\frac{1}{\alpha}\left[(1 + \kappa(1-\alpha))\left(1 + \frac{\beta_2}{\beta_1}\frac{1}{\kappa}\right) - (1-\alpha)(1+\kappa)(1-\alpha\beta_2)\right]
\end{aligned}$$

This is the solution for $\widetilde{\sigma}$, as show in the text. Using the definition of κ this expression can be written as: One can try to simplify this expression as follows

$$\widetilde{\sigma} = \frac{\beta_1}{\beta_1 - \beta_2}\frac{1}{\alpha}\frac{1}{\kappa}\left[(1 + \kappa(1-\alpha))\left(\kappa + \frac{\beta_2}{\beta_1}\right) - \kappa(1-\alpha)(1+\kappa)(1-\alpha\beta_2)\right]$$

or

$$\begin{aligned}
\widetilde{\sigma} &= \frac{\beta_1}{\beta_1 - \beta_2}\frac{1}{\alpha}\frac{1}{m} \\
&\quad \left[m^2\left(\frac{\beta_1}{\beta_2}\right)^{\frac{\alpha}{1-\alpha}} + m\left((1-\alpha)\frac{\beta_2}{\beta_1} + \alpha + \alpha(1-\alpha)\beta_2\right) + \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{1-\alpha}}(1-\alpha)\alpha\beta_2\right]
\end{aligned}$$

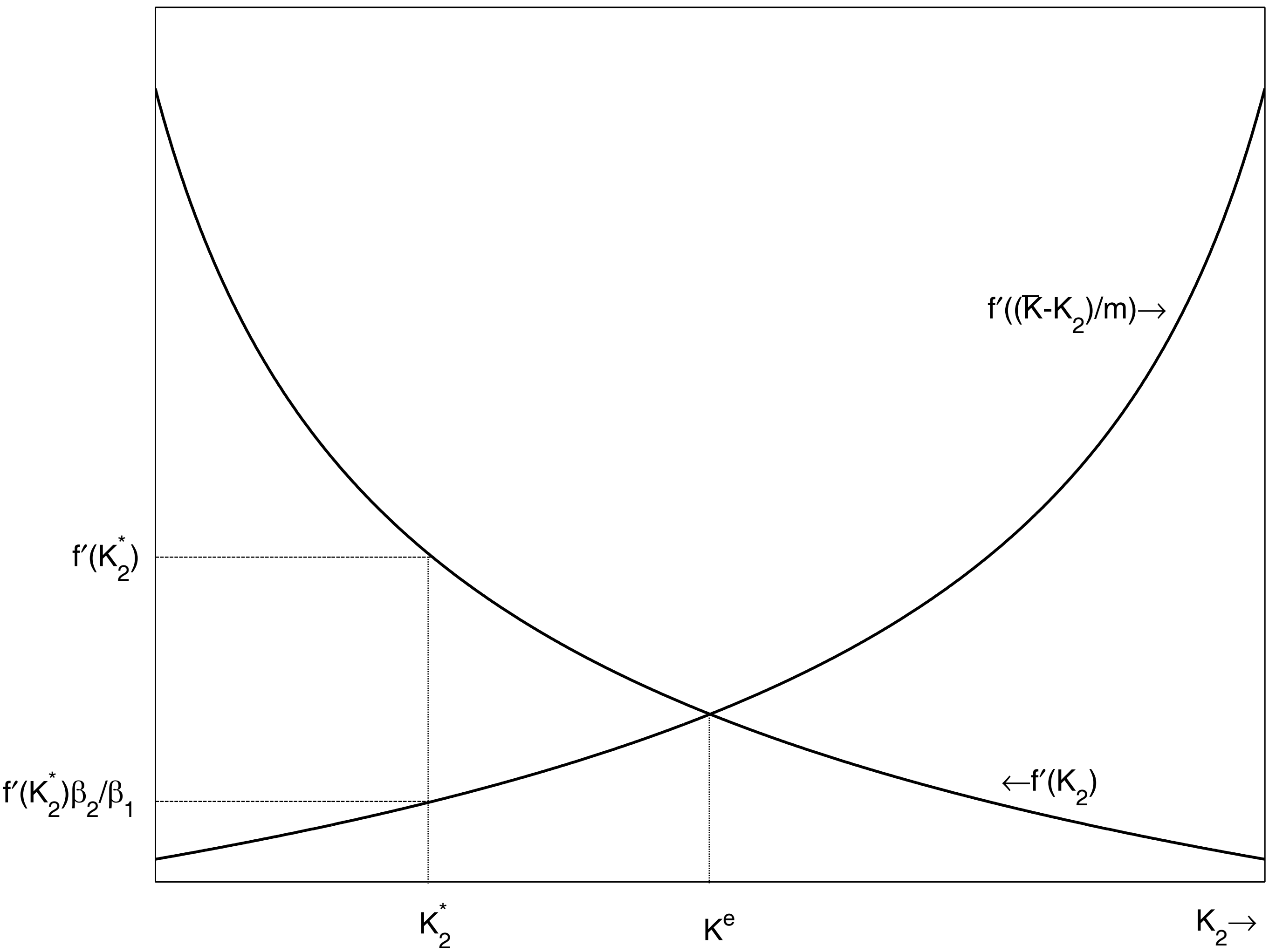


Figure 1: Steady State Distribution of Collateral

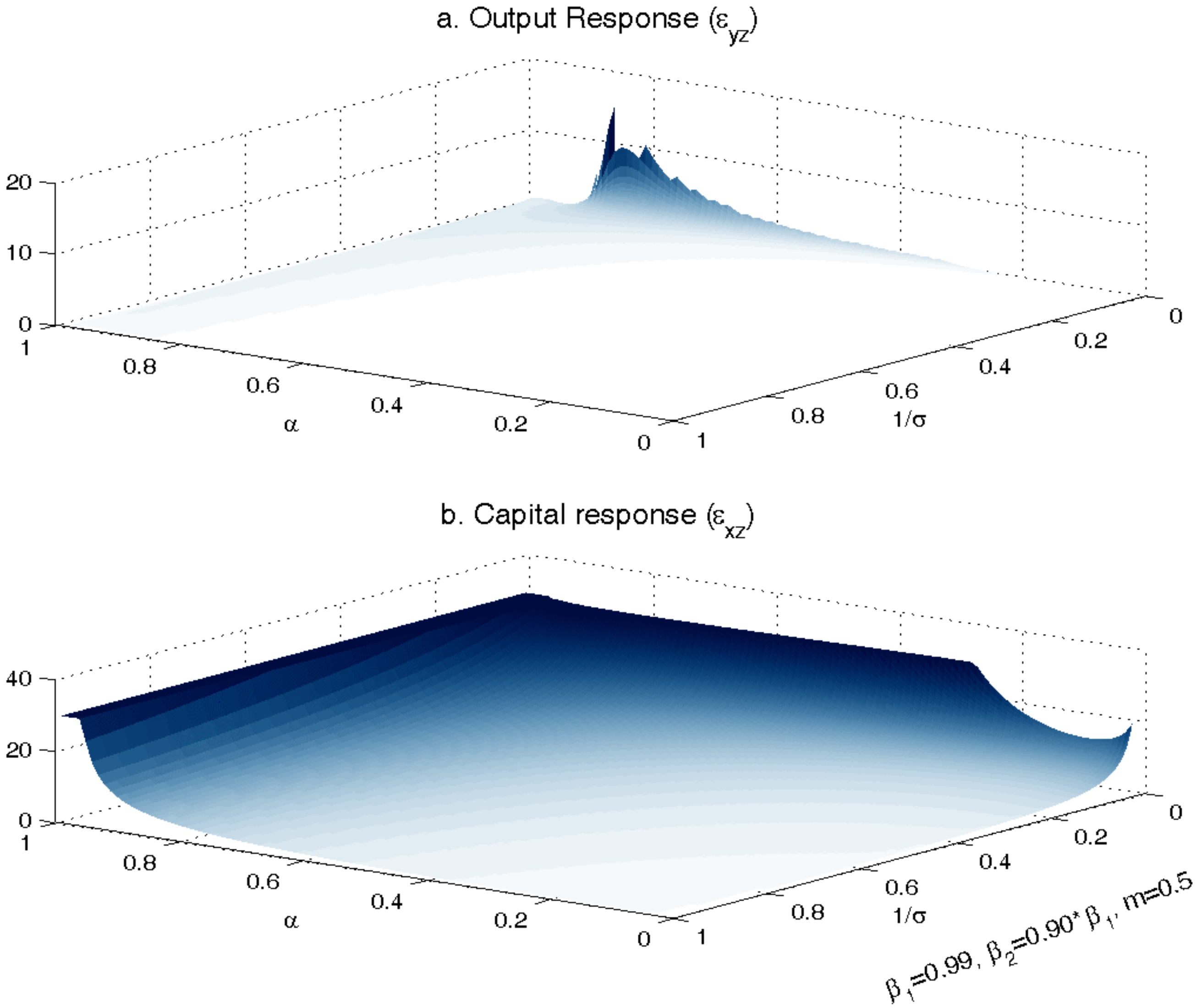


Figure 2: Response of Ouput and Collateral to an Unanticipated Shock

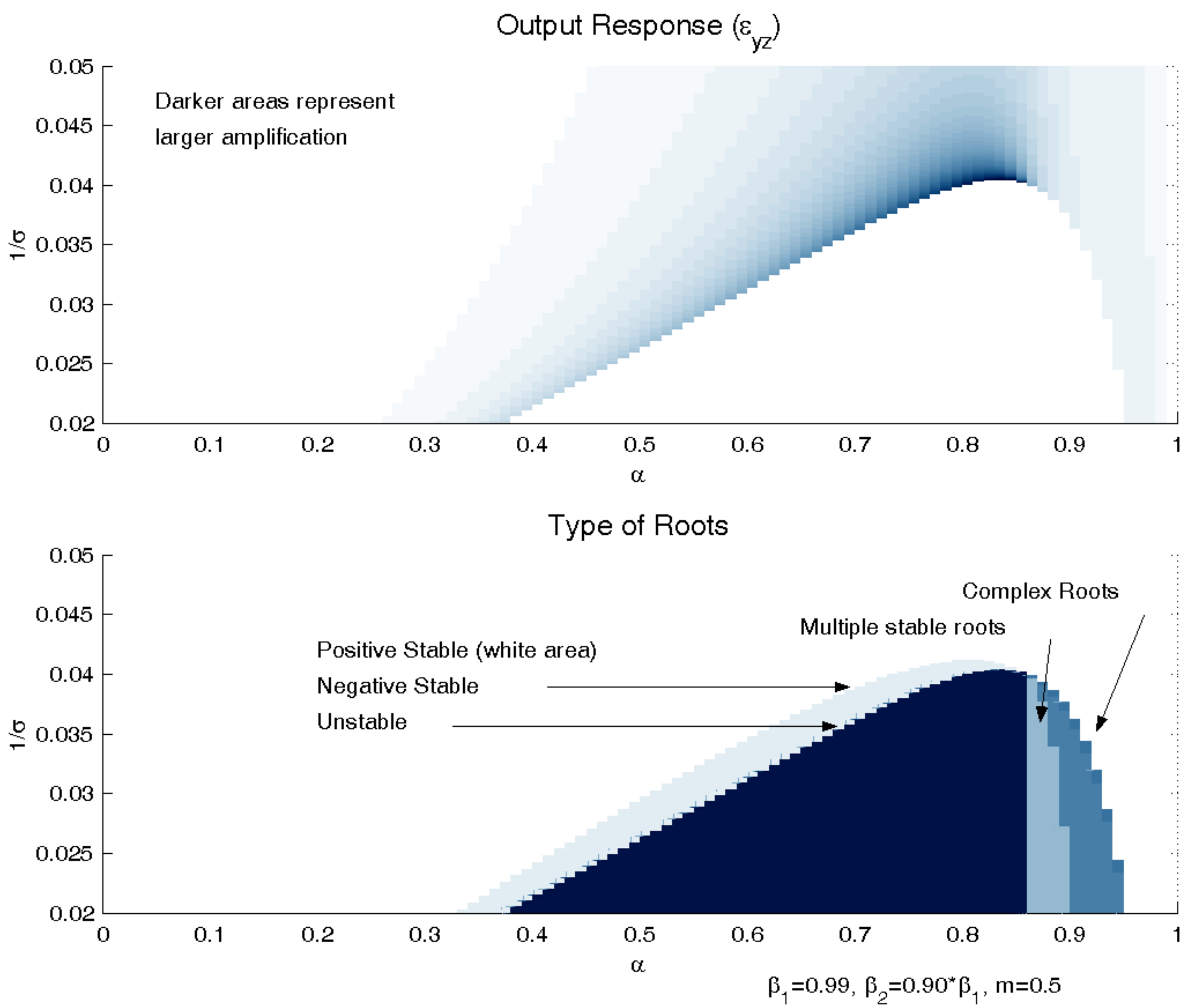


Figure 3: Output Response and Type of Roots

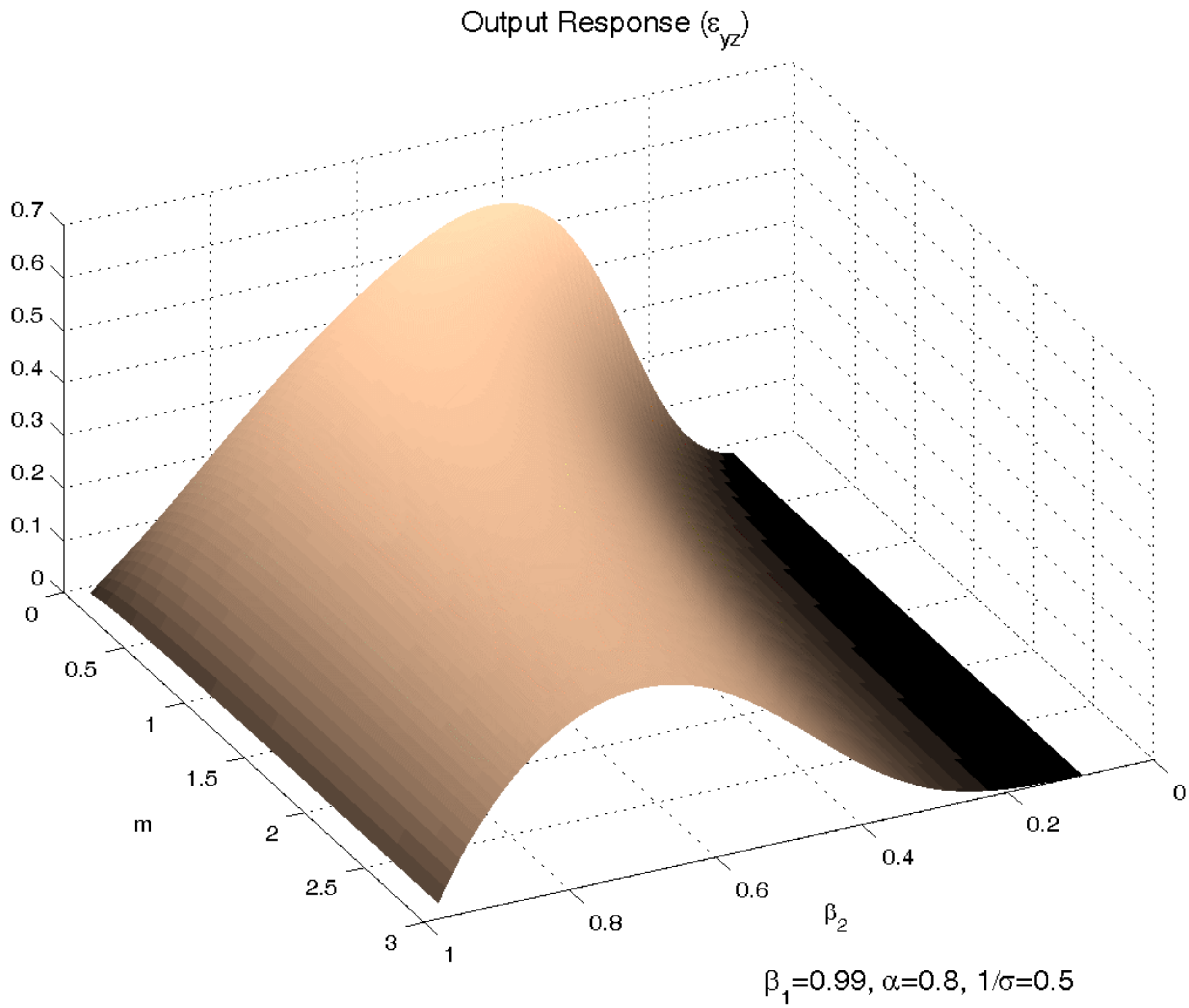


Figure 4: Output response as a function of m and the productivity ratio

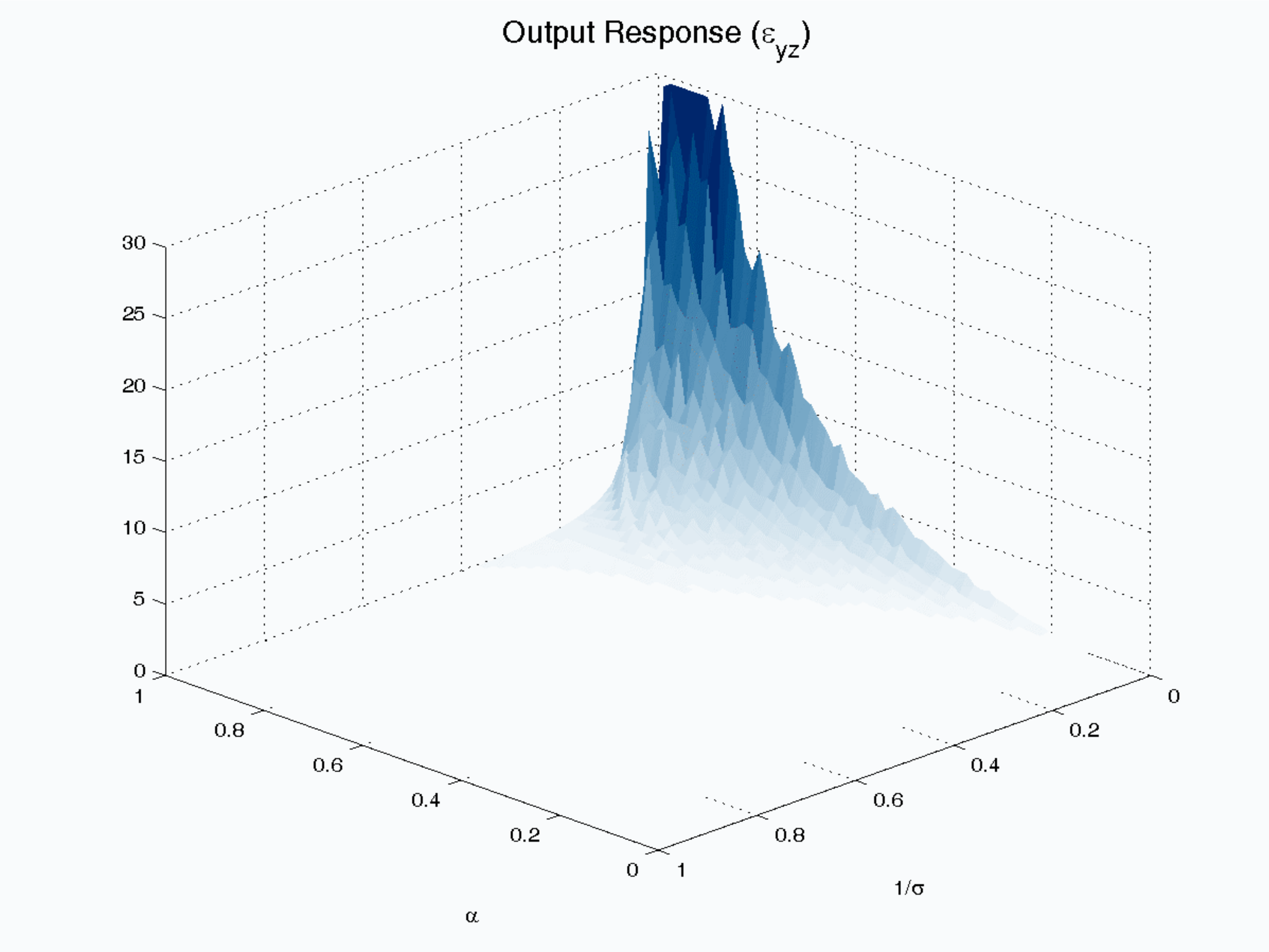
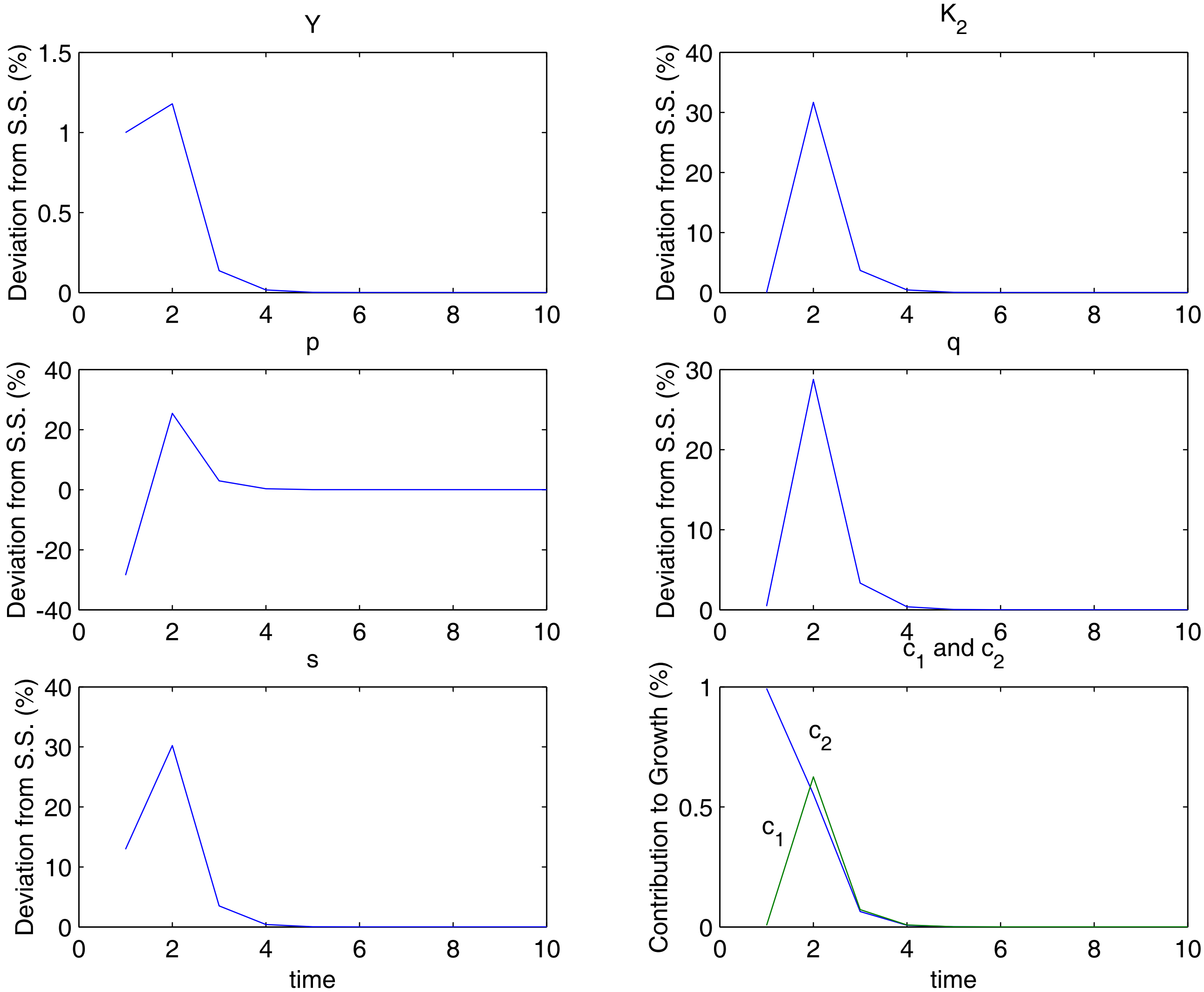


Figure 5: Output response for optimal choices of m and productivity differences



β_1=0.99, β_2=0.90*β_1, 1/σ=0.037, α=0.5, m=0.3

Figure 6: Impulse response functions to a 1% productivity shock

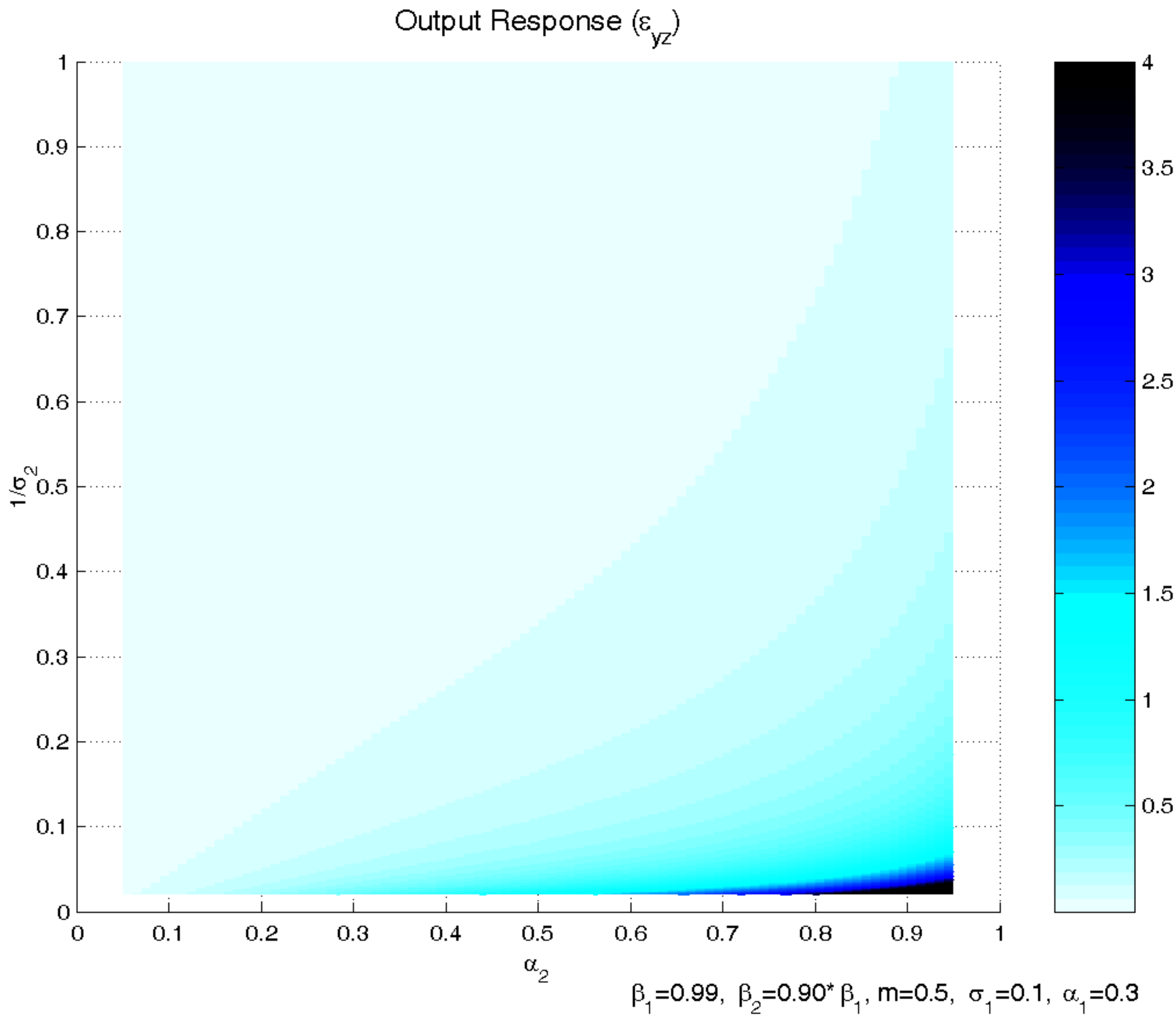


Figure 7: Output response under heterogenous preferences and technologies