Collateral Constraints in a Monetary Economy

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Abstract

The purpose of this paper is to analyze the role of collateral constraints as a transmission mechanism of monetary shocks. We do this by introducing money in the heterogeneous-agent real economy of Kiyotaki and Moore (1997). Money enters in a cash-in-advance constraint and is injected via open-market operations. In the model, a one-time exogenous monetary shock generates persistent movements in aggregate output, whose amplitude depends on the degree of debt indexation. Monetary expansions can trigger a large upward movement in output, while monetary contractions give rise to a smaller downward movement. This asymmetry occurs because full indexation of debt contracts can only be effective following a monetary contraction. In contrast, following a monetary expansion indexation can only be *partial* because debtors end up paying back just the market value of the collateral. Due to the existence of both cash-in-advance and collateral constraints, monetary shocks generate a highly persistent dampening *cycle* rather than a smoothly declining deviation.

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1 Introduction

The extent and mechanism through which monetary policy affects real economic activity over the business cycles has been a long-standing question in macroeconomics. Different mechanisms that explain the propagation of money shocks have been proposed. These include sticky prices, wage contracting, monetary misperceptions, and limited participation.[1] Another mechanism that has received special attention in recent years is credit-market imperfections. In particular, the agency-cost model of Bernanke and Gertler (1989) has been extended to monetary environments in order to analyze how fluctuations in borrowers' net worth can contribute to the amplification and persistence of exogenous money shocks to the economy.[2]

In contrast with these agency-costs models, little attention has been devoted to analyzing monetary economies in which agents face endogenous credit limits determined by the value of collateralized assets. The environment we have in mind is one in which lenders cannot force borrowers to repay their debts unless debts are secured. The use of this type of credit constraints appears to be a promising avenue to generate the amount of amplification and persistence lacking in current monetary models. This conjecture is motivated by the results obtained for the real-economy models of Kiyotaki and Moore (1997), Kiyotaki (1998) and Kocherlakota (2000) among others, who have shown that collateral constraints are a powerful mechanism of amplification and persistence of real shocks.[3] The central idea is that bad times for the economy are also times when the liquidation value of the collateral is low, as potential buyers face difficult times. This reduces debt capacity, which in turn reinforces the fall of the collateral price, as potential buyers become even more

[1]See Cooley and Hansen (1998) for an illustration of the role of monetary shocks in the equilibrium business cycle theory.

[2]See Fuerst (1995), Bernanke, Gertler and Gilchrist (1999), and Carlstrom and Fuerst (2000). Credit-market imperfections in these models emerge from asymmetric information and costly-state verification. In this framework, entrepreneurs borrow to pay the amount of the factor bill that is not covered by their net worth. Lenders must pay a monitoring cost in order to observe the entrepreneur's project outcome. If an entrepreneur has little net worth invested in the project, monitoring costs increase because there is larger divergence between the interests of the entrepreneur and the lender, and so the premium for external financing is larger. With procyclical net worth, periods of low output are associated with higher monitoring costs and a higher external finance premium. This mechanism amplifies the effects of external shocks on production and investment.

[3]Scheinkman and Weiss (1986) also study the effects of borrowing contraints in the presence of uninsurable risk. They simulate a lump-sum monetary injection that changes the distribution of assets across agents.

cash-strapped.

This paper analyzes the role of collateral constraints as a transmission mechanism of monetary shocks. We do this by introducing a cash-in-advance constraint for consumption and investment in the real-economy model of Kiyotaki and Moore (1997). We exploit the simplicity of this framework to study monetary injections carried out via open-market operations, as opposed to the less realistic but simpler helicopter drops employed by many monetary models. Due to the presence of credit-market imperfections, the exact path of the money supply is crucial to determine the real effects of open-market operations. We choose a parsimonious type of monetary paths which avoid changes in long-run inflation and fiscal variables. Thus, current monetary expansions need to be offset by future monetary contractions to avoid changes in inflation or unstable government-bond paths. In this monetary economy the price of the collateral plays a central role in generating large and persistent effects of exogenous shocks. Moreover, the response of the nominal interest rate becomes also crucial in determining the effects of shocks.[4].

The main finding of this paper is that a monetary shock can generate persistent movements in aggregate output, whose amplitude depends on the degree of debt indexation. In particular, the larger the indexation, the lower the amplitude of the fluctuations. This result follows from the fact that in our model, any redistribution of resources that favors borrowers is output enhancing because in equilibrium borrowers are more productive than lenders.

We also find that business cycles are asymmetric in the model. In particular, while monetary expansions can trigger a large upward movement in output, monetary contractions give rise to a smaller downward movement. This asymmetry occurs because full indexation of debt contracts can only be effective following a monetary contraction. In contrast, following a monetary expansion indexation can only be *partial* because in this case debtors repudiate their loans, there is debt renegotiation, and they end up paying back just the market value of the collateral. This is so because when there is a money expansion, the debt repayment increases by more than the market value of the collateral, and so debtors have incentives to repudiate their debts.

[4]In Kiyotaki and Moore (1997), Kiyotaki (1998) and Kocherlakota (2000) the interest rate is constant in equilibrium.

A third property of the model is that monetary shocks trigger highly persistent dampening *cycles* rather than smoothly declining deviations. This occurs due to the interplay between cash-in-advance and collateral constraints. In particular, if following an exogenous shock borrowers were able to acquire more capital, the full impact of the shock would be delayed because with a binding cash-in-advance constraint, collateral can only be accumulated gradually. The cyclical dynamics of the model is consistent with the hump-shaped pattern of output response to shocks that has been observed in the data.[5]

Finally, the model also generates endogenous limited participation in the government-bonds market due to the fact that in equilibrium, collateral constraints are binding only for a set of agents. This implies that only unconstrained agents hold government bonds and can participate in open-market operations. In this context, the propagation of the money shock is nontrivial because agents differ not only in whether they are or not credit constrained, but also in their productivity.

This paper offers a novel approach to the propagation of monetary shocks by combining collateral and cash-in-advance constraints, in a world where changes in money supply occur via open-market operations. As indicated above, other papers in the literature emphasize the role of credit-market imperfections based on the existence of agency costs, but do not explicitly consider collateral constraints. For instance, Bernanke, Gertler and Gilchrist (1999) embed the agency-cost model into a dynamic new-keynesian framework that incorporates money, monopolistic competition and nominal price rigidities. They find that credit-market frictions amplify and propagate shocks in a quantitatively significant way. However, in this paper loan contracts are not indexed to interest rate shocks, and it is not clear how important this nonindexation is in generating such amplification. Further, their model is fairly complicated because in order to replicate the hump-shaped behavior of output, they need to allow for lags in investment and differential credit access across firms.

On the other hand, Calstrom and Fuerst (2000) conclude that even though the agency-cost model delivers substantial propagation of monetary shocks, it does not deliver amplification. They introduce money into the real model of agency costs developed in Calstrom and Fuerst (1998).

[5] See Bernanke, Gertler and Gilchrist (1999), and Carlstrom and Fuerst (2000). We review these two papers below.

Money enters through a cash-in-advance constraint on household purchases of consumption and investment goods. Different from Bernanke and Gertler (1989), this paper models entrepreneurs as long lived. With long-lived borrowers, the net worth becomes a state variable which contributes to the persistence of shocks.[6]

Outside of the agency-cost literature, our paper is also related to Cooley and Quadrini (1998). They calibrate a monetary general equilibrium model with heterogeneous and long-lived firms where financial factors play an important role in production and investment decisions. Firms differ in size and face borrowing constraints. Small firms tend to rely more on external financing, and are more sensitive to monetary shocks. The response of the economy to monetary shocks is characterized by greater persistence than is typically found in other business cycle models. However, while these shocks have only a small impact on aggregate output, they lead to considerable volatility in financial markets. Since our model is much simpler than Cooley and Quadrini's, we are able to obtain analytical results and develop some intuition on the fundamental mechanisms that generate greater persistence of money shocks.

The reminder of the paper is organized as follows. Section 2 presents the model and characterizes the steady state. In Section 3 we discuss the dynamics of the model in response to a monetary shock. The dynamic structure of the model can be summarized by a nonhomogeneous second-order difference equation in the distribution of capital across agents. We parameterize the model and provide a numerical illustration of the dynamics in Section 4. Finally, Section 5 concludes. Technical details omitted in the text are presented in the Appendix.

2 The model

The model for this heterogeneous-agent economy is an extension of the framework of Kiyotaki and Moore (1997). We keep the main features of their model and introduce money using a cash-in-

[6] Calstrom and Fuerst (2000) consider two different ways of modeling entrepreneurs, and compare how the dynamics of the model change under each scenario. In one scenario entrepeneurs are infinitely lived but discount the future more heavily than households. In a second scenario, some entrepreneurs die each period and are replaced by new births so as to hold steady population. It turns out that when entrepreneurs are infinitely lived net worth responds more sharply to shocks.

advance (CIA) constraint. There are two goods in this economy: a durable asset (capital), and a nondurable commodity (output). We focus on the effects of monetary shocks on the distribution of capital across agents and abstract from capital accumulation. Capital is available in an aggregate fixed amount $\bar{K}$.

There are two types of private agents in this economy. They are both risk neutral, but operate different technologies and have distinct discount factors. As will become clear below, around the steady state the more patient agents become lenders, while the impatient agents become borrowers. To abbreviate, let us refer to the two types of agents as borrowers and lenders. Both types of agents face a CIA constraint and a collateral constraint. Finally, the government in this economy has the only role of controlling money supply through open-market operations.

Events in this model occur as follows. Assume that there are two identical members per household who carry out different activities. Households enter each period with money balances stored from the previous period. Production takes place overnight. Early in the morning households observe the money shock and borrowers repay their outstanding debts in output.[7] During the day, all markets are opened simultaneously. The first member of the household uses the money balances to make transactions in both the capital and goods markets. He can buy or sell capital, and buy goods.[8] The second member stays at home selling the goods the household has produced, making transactions in the money market and contracting new debt. Financial transactions must satisfy a standard budget constraint for the household, as well as a collateral constraint.

2.1 Borrowers

The measure of borrowers is normalized to one. Their technology is given by the production function $y_t = (a+c)k_{t-1}$, where k_{t-1} is their capital stock at the end of last period.[9] They choose sequences

[7]Borrowers repay their outstanding debts at the beginning of the period to ensure that if the debt is repudiated, lenders can appropriate the collateral. As in other CIA models, we assume that households value the different "types" of output produced by other households. This implies that when lenders get paid in output, they will sell it in exchange for money, and buy other varieties of output.

[8]Agents selling capital increase their money holdings and can use these balances to buy consumption good the same day.

[9]Thus, at any point of time total supply of output is completely predetermined by the distribution of the capital across the two types in the previous period.

of consumption $\{x_t\}$, capital holdings $\{k_t\}$, nominal money balances $\{m_t^d\}$, private issued bonds $\{b_t\}$, and government-bonds purchases $\{h_t\}$ to solve the following problem for given sequences of output prices $\{p_t\}$, nominal capital prices $\{q_t^n\}$, nominal interest rates $\{R_t\}$, and government-bonds nominal rates $\{R_t^h\}$

$$\max \sum_{t=0}^{\infty} \beta^t x_t$$

subject to

$$q_t^n(k_t - k_{t-1}) + p_t x_t \leq m_{t-1}^d, \tag{1}$$

$$m_t^d + R_t b_{t-1} + h_t \leq (a + c)p_t k_{t-1} + b_t + R_t^h h_{t-1}, \tag{2}$$

$$R_{t+1} b_t \leq q_{t+1}^n k_t, \tag{3}$$

where the nominal interest rate R_t is defined as the interest paid on loans made at $t - 1$. Equation (1) is the CIA constraint. Money is required for both consumption and investment. Equation (2) is the budget constraint. The revenues collected through output sales, new bonds issued, and the proceeds from government-bond holdings must be enough to accumulate new money balances, pay outstanding debt obligations, and purchase government bonds. Finally, equation (3) corresponds to the collateral constraint. Borrowing can only take place up to the point where the principal plus interest is secured by the market value of the capital owned by the household.

It is assumed that only the fraction a of the output is tradable between borrowers and lenders. The fraction c can be traded only among borrowers, and it can be interpreted as a subsistence minimum consumption. We refer to this fraction as the nontradable output. The purpose of the assumption is to avoid the situation in which borrowers continuously postpone consumption.[10]

In Appendix A we prove that around the steady state of the model the borrower's optimal plan is to consume only the nontradable fraction of output, i.e. $x_t = ck_{t-1}$, to borrow up to the

[10] Kiyotaki and Moore (1997) introduce a similar assumption. As will be explained later on, due to the linearity of preferences borrowers would like to continuosly postpone consumption in exchange for investment. This is avoided by introducing a nontradable fraction of output, which we think of as subsistence minimum consumption. Notice that money is required to buy nontradable output because this type of output can be traded among borrowers. One can think that households can only produce say fruit of a particular color, but they value fruits of all colors.

limit imposed by the collateral constraint, and to invest all remaining resources. This implies that borrowers do not purchase government bonds, i.e. $h_t = 0$, and that the CIA constraint is binding. These results hold under the following assumption

Assumption 1.

$$\frac{c}{a} > \frac{(1-\beta)}{\beta^2}\frac{(2-\beta-\beta')}{(1-\beta')},$$

where β' is the lenders discount factors. This condition is easy to satisfy if the discount factors are similar and close to 1.[11]

We can use equations (1), (2) and (3) to obtain

$$k_t = \frac{1}{u_t}\left[(a+q_t)k_{t-1} + \frac{1}{1+\pi_t}\frac{m^d_{t-1}}{p_{t-1}} - \frac{R_t}{1+\pi_t}\frac{b_{t-1}}{p_{t-1}} - \frac{m^d_t}{p_t}\right],$$

where $\pi_t \equiv \frac{p_t - p_{t-1}}{p_{t-1}}$ is the inflation between $t-1$ and t, and $q_t \equiv \frac{q^n_t}{p_t}$ is the real price of capital. The term in brackets corresponds to the real net worth of borrowers, which consists of the value of tradable output, plus the value of capital held from the previous period, plus the real money balances brought from the previous period, minus the real value of debt repayments, minus money balances reserved for next period's purchases. Finally, the users cost of capital for borrowers, u_t, is given by

$$u_t \equiv q_t - \frac{1+\pi_{t+1}}{R_{t+1}}q_{t+1}. \tag{4}$$

Thus, equation (??) says that borrowers use all their net worth to finance the difference between the value of their capital $q_t k_t$ and the amount they can borrow against each unit of capital $\frac{q_{t+1}}{R_{t+1}}(1+\pi_{t+1})k_t$ in real terms. Notice that borrowers discount the future value of the capital at the nominal interest rate. This is the case, as will become clear below, because in equilibrium borrowers need to borrow in order to buy capital.

[11] In this case, $\frac{(2-\beta-\beta')}{(1-\beta')}$ is some constant near to 2, and $\frac{(1-\beta)}{\beta^2}$ is close to zero. Further, in the proposed equilibrium $\frac{c}{a}$ is the ratio between the marginal propensity to consume and the marginal propensity to save for borrowers, which can be assumed to be bounded away from zero.

2.2 Lenders

The mass of lenders in the economy is n. Lenders differ from borrowers in their production technology and in the preferences. Lenders use a strictly concave technology, and they are more patient than borrowers. Their production function is given by $y_{t+1} = G(k'_t)$, where $G' > 0$, $G'' < 0$ and $G'(0) = \infty$. Lenders choose sequences of consumption $\{x'_t\}$, capital holdings $\{k'_t\}$, nominal money balances $\{m'_t\}$, bonds holdings $\{b'_t\}$, and government-bonds purchases $\{h'_t\}$, to solve the following problem for given sequences of output prices, nominal interest rates, and nominal capital prices

$$\max \sum_{t=0} \beta'^t x'_t$$

subject to

$$q^n_t(k'_t - k'_{t-1}) + p_t x'_t \le m^{d\prime}_{t-1}, \tag{5}$$

$$m'_t + R_t b'_{t-1} + h'_t \le p_t G(k'_{t-1}) + b'_t + R^h_t h'_{t-1}, \tag{6}$$

where the prime denotes a lender's decision variable. Lenders face a CIA constraint and a budget constraint. We do not explicitly write a collateral constraint for these agents. Around the steady state this constraint is not binding due to the fact that lenders determine the interest rate in the economy, and therefore face an interior solution in bonds. In order to obtain this result, it is assumed that lenders have a larger discount factor than borrowers.

Assumption 2. $\beta' > \beta$.

Let $\beta'^t \Omega_t$ be the Lagrange multiplier associated to the CIA constraint and $\beta'^t \Lambda_t$ the one for the budget constraint. Then, the first order optimality conditions for the problem above are given by[12]

$$x'_t : 1 = \Omega_t p_t,$$

$$m'_t : \Lambda_t = \beta' \Omega_{t+1},$$

[12]These are the first order conditions for interior solutions. Assumptions above guarantee such result.

$$b'_t : \Lambda_t = \beta' R_{t+1} \Lambda_{t+1},$$

$$h'_t : \Lambda_t = \beta' R^h_{t+1} \Lambda_{t+1},$$

$$k'_t : q^n_t \Omega_t - \beta' q^n_{t+1} \Omega_{t+1} = \beta' \Lambda_{t+1} p_{t+1} G'(k'_t).$$

It is immediate that the following arbitrage condition holds: $R_t = R^h_t$. From the optimality conditions above is easy to obtain expressions for the equilibrium nominal interest rate and the users cost of capital for the lenders u'

$$R_t = \frac{1}{\beta'} \frac{p_{t+1}}{p_t} \equiv \frac{1 + \pi_{t+1}}{\beta'}, \tag{7}$$

$$u'_t \equiv q_t - \beta' q_{t+1} = \frac{\beta'^2}{1 + \pi_{t+2}} G'(k'_t). \tag{8}$$

Notice that the equilibrium nominal interest rate R_t depends on the inflation rate at $t+1$. This is simply due to the CIA constraint. Loans made at $t-1$ are repaid at time t in the bonds market. However, lenders can only spend the returns of the loan at time $t+1$, due to the CIA constraint. Thus, lenders must be compensated for any inflation at time $t+1$.

Equation (8) states that lenders equate their users cost of capital with the present value of its marginal product. Since in equilibrium these agents are not credit constrained, the users cost is simply the difference between the cost of buying capital today and the discounted value of selling capital tomorrow. Notice that, in contrast with the borrowers, the lenders' users cost is not affected by inflation since the proceeds of selling the capital can be consumed or invested immediately, without requiring previous accumulation of cash.

2.3 Government

The government controls money supply in this economy through open-market operations (OMOs), which take place in the bonds market. Let H^s_t be the nominal supply of government-issued bonds.

The stock of money supply M_t^s in this economy is given by

$$M_t^s = M_{t-1}^s - H_t^s + R_t H_{t-1}^s,$$

where

$$H_t^s = \tau H_{t-1}^s,$$

so that at time t the government withdraws an amount τH_{t-1}^s of money and injects $R_t H_{t-1}^s$ back into the economy. There are two comments in order. First, we choose a simple law of motion for government bonds H_t^s. This simplicity is convenient for our purpose of analyzing the effects of a one-time money shock. Notice that following this shock, unless $\tau < 1$ for all t, government bonds may exhibit an explosive path. To avoid this, any one-time money expansion through OMOs must be eventually followed by a "policy reversal" or "sterilization" that guarantees convergence back to the steady state. In particular, the size of τ determines the speed at which such monetary contraction takes place. We are aware that since credit markets are imperfect in this economy, real effects of monetary shocks depend on the path of government debt. Although we choose a parsimonious law of motion for H_t^s, we will discuss below the role of the size of τ in our results, as well as other paths for government debt.

Second, notice that we do not consider a rebate of the inflationary tax. Since some agents face corner solutions, such rebate cannot be lump-sum in general. For example, simple helicopter drops redistribute wealth, and affect agents decisions. Since here we want to focus on the effects of the "pure monetary shock", we do not include any rebates in the model. Tax rebates in fact may reinforce the results of the paper.[13]

[13] The intuition for this result is simple. Suppose the economy starts off at the steady state and there is a one-time money expansion. Assume that borrowers were to receive a money transfer that compensates them for the inflationary tax in an amount higher than their optimal consumption. This may happen, for example, with helicopter drops. In this case, borrowers will buy capital with the extra resources, and next period output would increase. This reinforces our results because, as will be shown below, in this economy monetary expansions generate booms. More details on this are available from the authors upon request.

2.4 Aggregate resource constraints

Let K_t, K_t', B_t, B_t', H_t, H_t', M_t^d, $M_t^{d\prime}$ be the aggregate variables corresponding to the lowercase individual variables. There are five markets in the model: consumption goods, capital, money, private bonds, and public bonds. By Walras' Law one needs only to consider four of them. The equilibrium conditions to clear the last four markets are

$$M_t^s = M_t^d + M_t^{d\prime} = m_t^d + nm_t^{d\prime},$$

$$B_t = b_t = -B_t' = -nb_t',$$

$$\bar{K} = K_t + K_t' = k_t + nk_t',$$

and since $H_t = 0$,

$$H_t' = nh_t' = H_t^s.$$

Using the market clearing conditions above along with equations (6) and (2) we obtain

$$M_t^s + H_t^s - R_t H_{t-1}^s \equiv M_{t-1}^s = p_t \left[(a+c)K_{t-1} + nG\left(\frac{\overline{K} - K_{t-1}}{n} \right) \right], \tag{9}$$

which is just the quantity equation.

2.5 Steady state

Define a steady state where all real variables are constant, and all nominal variables grow at the constant rate π, which is the steady-state growth rate of money supply. From the law of motion of government bonds it follows that to keep government's debt $\frac{H^s}{p}$ constant in real terms, it must be the case that $\tau = 1 + \pi$. Thus, if $\tau > 1$ then $\pi > 0$, while for $\tau \leq 1$, $\pi = 0$.

Let d be the steady-state government bonds to money supply ratio, i.e., $d \equiv \frac{H^s}{M^s}$ which can also be seen as the public debt to money ratio. Using the law of motion of money supply and the fact that the steady-state nominal interest rate is $R = \frac{1+\pi}{\beta'}$ we obtain

$$d \equiv \frac{H^s}{M^s} = \frac{\pi\beta'}{(1+\pi)\left(1-\beta'\right)},$$

which implies that when $\pi = 0$, then $H^s = 0$.

Next, it is easy to see that the steady-state users cost of capital for lenders and borrowers is the same: $u = u' = q(1-\beta')$. Further, since under the proposed equilibrium the collateral constraint (3) binds for the borrowers, we can use R, u' and the budget constraint of these agents (2) to get: $u = a + c - \frac{M^d}{pK^*}$, where K^* is the borrowers' steady-state capital level. Next, using the CIA constraint (1) one obtains: $\frac{M^d}{p} = cK^*(1+\pi)$, i.e. borrowers' real money balances exactly cover their consumption adjusted by inflation.

Combining the last two expressions we obtain: $u = a - \pi c$. Notice that if $\pi = 0$, we obtain the intuitive results that $\frac{M^d}{p} = cK^*$, and $u' = u = a$. This last equation means in a steady state with no money growth, the users cost equals the tradable marginal product of capital.

Finally, using equation (8), we obtain an implicit solution K^*

$$G'\left(\frac{\overline{K} - K^*}{n}\right) = \frac{1+\pi}{(\beta')^2}(a - \pi c) \tag{10}$$

The equation above, along with Assumption 1 imply that in equilibrium borrowers have higher marginal product of capital than lenders.

The following proposition summarizes the main features of the steady state.

Propostion 1. *Under Assumptions 1 and 2,*

(i) if $G'\left(\overline{K}/n\right) < \frac{1+\pi}{(\beta')^2}(a - \pi c)$ *there exists a unique steady state;*

(ii) $\frac{\partial K^*}{\partial \pi} \neq 0$ for $(1+2\pi)c \neq a$, *so that inflation affects the steady-state output* Y^*.

Proof: The existence of a unique steady state level K^* is guaranteed from the properties of the production function $G(.)$. It is easy to see that the left-hand side of equation (10) is continuous and strictly increasing in K, while the right-hand side is a constant. If $G'\left(\overline{K}/n\right) <$

$\frac{1+\pi}{(\beta')^2}(a-\pi c)$ the left and right-hand side cross only once. Figure 1 illustrates the determination of the steady state. The second property follows easily. ■

It is interesting that in the long run money is not superneutral as indicated by Proposition 1, *(ii)*. The intuition for this result is as follows. Inflation acts as a tax for all agents, but in the margin it affects differently borrowers and lenders. Higher inflation decreases the marginal cost of investing for both types, i.e. it decreases the users cost of capital. For a given K^*, borrowers net worth decreases with higher inflation because they must demand more money to sustain their consumption, cK^*. Further, since borrowers are credit constrained, they are in a corner solution. In contrast, lenders have an interior solution and since u has decreased, their marginal benefit of investing needs to decrease, which can only happen if lenders' capital holdings, $\overline{K}-K^*$, increase. Thus, money is not superneutral due to the asymmetric effect of inflation on constrained and unconstrained agents.[14]

3 Dynamics

To simplify the analysis, we only present the dynamics of the model around the steady state, and assume zero steady-state inflation, $\pi = 0$. The solution for the case $\pi > 0$ is summarized in Appendix D. We also assume that β' is close to 1. This occurs, for example, if the length of the periods is small. This assumption allows as to obtain some sharp analytical results, but numerical simulations confirm that the main results hold even if β' is far from 1. Let $g_t \equiv \frac{M_t^s}{M_{t-1}^s}$, i.e. g_t is one plus growth rate of money supply, $v_t \equiv \frac{p_t}{p_{t-1}}$, i.e. v_t is one plus the inflation rate. Thus, in the steady state, $g = v = 1+\pi$. In general, let $\hat{x}_t = \frac{x_t - x^*}{x^*}$ denote the rate of deviation of a variable x from its steady state value.

[14] If the borrowers' propensity to consume $\frac{c}{a+c}$ is larger than 0.5 then higher inflation reduces output, a result consistent with Abel (1984). However, if money is injected via helicopter drops rather than via OMOs, higher steady-state inflation may have the opposite results, i.e. higher π implies larger K^* and larger Y^*. This occurs if borrowers receive a fraction of the transfer higher than their steady-state consumption share, $\alpha \equiv \frac{cK^*}{Y^*}$. In this case, borrowers are overcompensated for the inflationary tax and, as a result, they can afford to buy additional capital with the extra resources. In addition, inflation increases the marginal cost of investing, u, but lenders are particularly hurt because they face and interior solution.

Assume that the economy starts off at the steady state, and that an unexpected one-time decrease in the growth rate of money $\varepsilon < 0$ occurs at $t = 0$, i.e. $\hat{g}_0 = \frac{\varepsilon}{1+\pi}$. Since the monetary contraction occurs through OMOs, H_0 increases above its steady state level ($H_0 > 0$). According with the law of motion for government bonds, $H_t^s = \tau H_{t-1}^s$, H_t gradually returns to zero to avoid changes in the long term inflation rate. Thus, the one-time money contraction at $t = 0$ is followed by a monetary expansion, i.e. by a "sterilization policy". In particular, the size of $\tau < 1$ determines the speed at which such monetary expansion takes place.

Using the law of motion of money supply and bonds, one can obtain the following path of money growth[15]

$$\hat{g}_0 = -\partial d_0,$$

and

$$\hat{g}_t = -(R-\tau)\tau^{t-1}\hat{g}_0.$$

Notice that this path is fully determined by the exogenous initial shock, and converges to zero at a rate determined by the size of τ. In particular, a larger τ implies a smoother sterilization of the monetary contraction.

To complete the characterization of the dynamics of the model, we need to solve for the paths of $\hat{v}_t$, $\hat{q}_t$ and $\hat{R}_t$. Linearizing equation (9) yields

$$\hat{v}_t = \hat{g}_{t-1} - \rho\left(\hat{R}_{t-1} - \hat{R}_{t-2}\right),$$

where $\rho = (a + c - G')\frac{K^*}{M^s/p}$. Notice that $\hat{v}_0 = 0$ because both output and the money supply used

[15] Here we compute the absolute deviations of the government-debt to money ratio ∂d_0 instead of the percentage deviations from the steady state because $d = 0$. The first expresion follows from linearizing the stationary version of the equation $M_0^s = M_{-1}^s - H_0^s$. For $t > 1$, combine the law of motion for H_t^s and M_t^s, transform variables to render them stationary, and linearize to obtain $\hat{g}_t = (R-\tau)\partial d_{t-1}$. Next, use the law of motion of government debt to obtain $\partial d_t = \tau\partial d_{t-1} = \tau^t\partial d_0$, which together with the previous expresion implies that $\hat{g}_t = (R-\tau)\tau^{t-1}\partial d_0 = -(R-\tau)\tau^{t-1}\hat{g}_0$.

for transactions in the goods market are predetermined. Next, linearizing equation (8) we obtain

$$\hat{q}_t - \beta' \hat{q}_{t+1} = \left(1-\beta'\right)\left(\frac{1}{\eta} - \rho\right)\hat{K}_t + \left(1-\beta'\right)\rho \hat{K}_{t+1} - (1-\beta')\hat{y}_{t+1},$$

where $\frac{1}{\eta} = -\frac{G'' K^*}{nG'} > 0$.[16] The equation above describes the forward-looking nature of capital prices, i.e. the price of capital at $t = 0$ depends on the whole path of capital distributions across types.

Finally, using the three expressions above, as well as the linearized versions of equations (1), (2) and (3), it is easy to show that $\hat{K}_t$ satisfies the following non-homogeneous second order difference equation for $t > 2$

$$\theta_0 \hat{K}_t = \theta_1 \hat{K}_{t-1} + \theta_2 \hat{K}_{t-2} + \mu_0 \tau^{t-2} \hat{y}_0, \tag{11}$$

where θ_0, θ_1, θ_2,and μ_0 are constants that depend on steady-state variables (see Appendix B). It can be shown that for β' close to 1, these constants are given by: $\theta_0 = 1 - \rho > 0$, $\theta_1 \approx 2\theta_0$, $\theta_2 \approx -\theta_0$, and $\mu_0 \approx -(1-\tau)^2$. This last term reflects that a money injection at $t = 0$ generates a negative trend in K_t as a result of the sterilization that takes place after the injection.

The previous equation summarizes the equilibrium dynamics of the model. It can be shown that $\hat{K}_t$ exhibits persistent and dampening cycles, as summarized in the following proposition

Proposition 2. *For β' sufficiently close to 1 and $\pi = 0$,*

(i) the general solution to (11) is

$$\hat{K}_t = Ar^t \cos(\omega t - \phi) + A_\tau \tau^t \hat{y}_0 \tag{12}$$

where A and ϕ are constants, $r = \sqrt{-\theta_2/\theta_0}$, $\omega = \cos^{-1}\left(\frac{\theta_1/\theta_0}{2r}\right)$, and $A_\tau = \frac{\mu_0}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}$.[17]

(ii) r is close to, but less than, 1, and ω is close to, but larger than, zero.

[16] The term $\frac{1}{\eta}$ can be rewritten as: $\frac{1}{\eta} = -\frac{G''(\overline{K}-K^*)/n}{G'} \frac{K^*}{\overline{K}-K^*}$, and so it can be interpreted as a measure of the elasticity of the marginal product of borrowers' capital, weighted by the ratio of borrowers to lenders' capital in the steady state.

[17] Note that $\lim_{\beta' \to 1} A_\tau = -\frac{1}{(1-\rho)}$.

Proof: See Appendix B. ■

Corollary. $\hat{k}_t$ *exhibits persistent and dampening cycles.*

To fully characterize the equilibrium solution, we require two additional conditions on the trajectory of $\hat{k}_t$. For reasons that we explain briefly, monetary injections via OMOs imply $\hat{k}_0 = 0$. Thus, the equilibrium path of the distribution of capital can be completely characterized in terms of $\hat{k}_1$. Using these two conditions, we obtain

$$A = \frac{\hat{k}_1 - A_\tau \tau \hat{b}_0}{\cos(\omega - \phi)} \text{ and } \cos(\phi) = -\frac{A_\tau \hat{b}_0}{A}.$$

Before proceeding to study the effects of a monetary shock, $\hat{b}_0$, it is useful to analyze first the simpler case of a real shock. In particular, suppose $\hat{b}_0 = 0$, but $\hat{k}_1 > 0$. This corresponds to an exogenous redistribution of capital. This exercise illustrates that our monetary model retains the powerful amplification mechanism displayed by the the real-economy version of Kiyotaki and Moore (1997). The following lemma presents this result

Lemma. $\hat{k}_t$ *attains its maximum at* $t^* = \frac{\phi}{\omega}$. *In addition,* $\lim_{\beta' \to 1} \frac{\hat{k}_{t^*}}{\hat{k}_1} = \infty$.

Proof: When $\hat{b}_0 = 0$, then $A = \frac{\hat{k}_1}{\cos(\omega - \phi)}$, $\cos(-\phi) = 0$, and $\hat{k}_t = \frac{\hat{k}_1}{\cos(\omega - \phi)} r^t \cos(\omega t - \phi)$. Therefore, $\hat{k}_t$ attains its maximum when $\cos(\omega t^* - \phi) = 1$, or $t^* = \phi/\omega$. Thus,

$$\hat{k}_{t^*} = \frac{\hat{k}_1}{\cos(\omega - \phi)} r^{\phi/\omega}$$

We have previously noticed that $\lim_{\beta' \to 1} \omega = 0$, and $\lim_{\beta' \to 1} r = 1$. Thus, in order to show that $\lim_{\beta' \to 1} \frac{\hat{k}_{t^*}}{\hat{k}_1} = \infty$, we only need to prove that $\lim_{\beta' \to 1} r^{\phi/\omega} = 1$, or equivalently, that , $\lim_{\beta' \to 1} \frac{\ln r}{\omega} = 0$. This can be easily shown by using L'Hopital rule, and the definitions of r and ω.

Let us turn now to the monetary shock. As we already mentioned in this case $\hat{k}_0 = 0$. This result follows from the following four facts: *i)* the money contraction occurs in the bonds market; *ii)* the shopper's only resources are the money balances accumulated during the previous period and the land holdings; *iii)* borrowers' consumption is predetermined, and as a consequence, lenders consumption is also predetermined; *iv)* the nominal price of consumption at the moment of the shock does not change. These facts together imply that at the moment of the shock households cannot change their investment level.

We now solve for K_1 following a monetary shock at $t = 0$. For this purpose, combine (1), (2), and (3) to obtain

$$q_1(K_1 - K_0) + cK_0 = \frac{a+c}{1+\pi_1}K_{-1} + \frac{1}{1+\pi_1}\frac{B_0}{p_0} - \frac{R_0}{(1+\pi_1)(1+\pi_0)}\frac{B_{-1}}{p_{-1}}, \tag{13}$$

where $\frac{B_{-1}}{p_{-1}}$ is the aggregate steady-state level of debt in real terms, and K_{-1} corresponds to the borrowers' steady-state capital level. We consider two relevant cases at this point: in one debt is fully indexed, while in the other debt can only be *partially* indexed. A fully-indexed contract states that borrowers must compensate lenders for any unexpected inflation. Thus, debt repayments at time zero are immune to period one's inflation π_1, i.e., $R_0 B_{-1} = \frac{1+\pi_1}{\beta'} B_{-1}$. Such contract is feasible in our model only if such repayments are lower than the value of the collateral, i.e., $\frac{1+\pi_1}{\beta'}\frac{B_{-1}}{p_{-1}} \leq \frac{q_0^n}{p_{-1}}K_{-1} = \frac{q_0^n}{p_0}K_{-1} = q_0 K_{-1}$. Otherwise, borrowers will repudiate the contract, and will be able to renegotiate the debt down to the market value of the collateral.[18] This is the case we call *partial indexation* of debt contracts because following a monetary shock, debt can only be indexed up to the market value of the collateral. We now discuss these two cases separately.

[18] The model of debt implicit in our model is the same as in Kiyotaki and Moore (1997). Collateral constraints arise in this economy because of the following two assumptions. The first is that once a borrower has started to produce with capital K_t, he is the only one with the skill to complete production in period $t+1$. The second is that the borrower's human capital is inalienable. These assumptions guarantee that if a borrower ever repudiates his debt contract, then he is able to renegotiate the debt down to the market value of the collateral. See Kiyotaki and Moore (1997), page 217.

3.1 Partial indexation

When debt is partially indexed, R_0 in (13) is not the equilibrium value. Rather, the term $R_0 \frac{B_{-1}}{p_{-1}}$ is replaced by the market value of the collateral $q_0 K_{-1}$. Linearizing equation (13) in this case yields

$$\hat{K}_1 = \frac{1}{1-\beta'\rho}\left[\beta'\hat{q}_1 - \hat{q}_0 + \left(1 - r^h\right)\varepsilon + \beta'(R-\tau)\hat{g}_0\right].$$

It turns out that in this case, the solution for $\hat{K}_1$ is simple[19]

$$\hat{K}_1 = \frac{1}{\theta_0}\left[(1-r^h) - \left(1-2\beta'\right)(R-\tau)\right]\hat{g}_0. \tag{14}$$

Since the expression above is algebraically simple, we can use it to analyze whether following the one-time monetary expansion in period $t = 0$, it is the case that $\hat{K}_1 > 0$ and so $\hat{Y}_2 > 0$. Further, if $\hat{K}_2 > \hat{K}_1$, since the model exhibits persistent dampening cycles, we should observe a boom in the economic activity as borrowers' capital level increases. Proposition 3 summarizes the conditions under which these results hold. Let $\alpha \equiv \frac{cK^*}{Y^*} < 1$ be the fraction of steady-state output consumed by the borrowers.

Proposition 3. *For β' sufficiently close to 1 and $\pi = 0$,*

(i) following a one-time increase in the money growth rate $\varepsilon > 0$ at $t = 0$, borrowers increase their capital holdings in period $t = 1$, i.e. $\hat{K}_1 > 0$. Further, the lower τ, the larger $\hat{K}_1$ is.

(ii) if τ is sufficiently close to 1 then $\hat{K}_2 > \hat{K}_1$, while if $\tau \to 0$ then a sufficient condition for $\hat{K}_2 > \hat{K}_1$ is that $\alpha > \frac{1}{3}$.

Proof: (i) When $\beta' \to 1$ it is the case that $\rho \to \alpha$ and that $r^h \to 0$. Then, $\theta_0 \to (1-\alpha)$. Thus, when $\beta' \to 1$ from equation (14) we have that: $\hat{K}_1 \to \frac{2-\tau}{1-\alpha}\hat{g}_0$, and since $\tau < 1$ and $\hat{g}_0 > 0$ it follows that $\hat{K}_1 > 0$. Notice that the more slowly government debt returns to the steady state, i.e. the larger τ, the lower the multiplier of monetary policy in the first period.

[19] Under partial indexation, the solution for $\hat{K}_1$ is the same as that implied by the non-homogeneous second order differential equation for $t = 1$ and $\hat{K}_0 = 0$.

(ii) From equation (11) we have: $\hat{K}_2 = \frac{\theta_1}{\theta_0}\hat{K}_1 + \frac{\mu_0}{\theta_0}\hat{g}_0$, and since when $\beta' \to 1$ we have that $R \to 1$ and so $\mu_0 \to -(1-\tau)^2$, then: $\hat{K}_2 \to \frac{1}{1-\alpha}\left[\frac{2-\tau}{1-\alpha} - (1-\tau)^2\right]\hat{g}_0$. If $\tau \to 1$, then $\hat{K}_2 \to \frac{1}{1-\alpha}\hat{K}_1$ and so $\hat{K}_2 > \hat{K}_1$. On the other hand, if $\tau \to 0$, then $\hat{K}_2 \to \left[\frac{1}{1-\alpha} - \frac{1}{2}\right]\hat{K}_1$, so that $\hat{K}_2 > \hat{K}_1$ if $\alpha > \frac{1}{3}$. ■

It is surprising that in this model, a monetary expansion generates a boom in output. Since due to the endogenous limited participation lenders are the ones handing in the cash to the government, one may guess that output should decrease. However, as shown in Proposition 3, this does not happen. It is interesting to highlight the mechanisms behind this result. For this analysis, it is useful to rewrite equation (13) as

$$q_1(K_1 - K_0) + cK_0 = \frac{a+c}{1+\pi_1}K_{-1} + \frac{\beta' q_1 K_0}{1+\pi_2} - \frac{R_0}{(1+\pi_1)(1+\pi_0)}\frac{B_{-1}}{p_{-1}},$$

where the left-hand side represents consumption and investment in $t = 1$, and the right-hand side are the real balances brought from period $t = 0$. In particular, the first term on the right-hand side are output sales; the second term is new debt contracted in $t = 0$; and the third term is the repayment for debt contracted in $t = -1$. Notice that consumption in $t = 1$ is fixed because K_0 remains at the steady-state level. Thus, the only way borrowers increase their investment in capital K_1, is if the right-hand side of the equation is large than its steady-state value.

First, notice that due to the monetary expansion $\hat{g}_0 > 0$, the level of prices in $t = 1$ increases, so that π_1 increases ($\hat{\pi}_1 > 0$). This hurts the borrowers because the first term on the right-hand side $\frac{a+c}{1+\pi_1}K_{-1}$ decreases. However, this decrease is more than compensated by an increase in $\frac{\beta' q_1 K_0}{1+\pi_2}$. This term increases because π_2 decreases ($\hat{\pi}_2 < 0$). In fact, in the general equilibrium of the model, an increase in the borrowers' capital level K_1 is consistent with a decrease in π_2. This is so because when borrowers increase their capital, then in the following period output increases ($\hat{Y}_2 > 0$), and prices decrease ($\hat{\pi}_2 < 0$). Further, this decrease in prices can be reinforced if the government starts reversing the monetary expansion, i.e. if $\hat{g}_1 < 0$. In fact, as indicated by part *(i)* in Proposition 3, a lower τ implies a higher $\hat{K}_1$, precisely because a lower τ is equivalent to a larger money contraction in $t = 1$ and a larger price decrease in $t = 2$.

Finally, since the third term on the right-hand side represents the repayment for debt contracted in $t = -1$, it indicates the role of debt indexation. In particular, under full indexation, since $R_0 = \frac{1+\pi_1}{\beta'}$, then this term would remain at its steady-state value. However, since under partial indexation R_0 increases by less than the increase in $1 + \pi_1$, then this term would decrease. This represents a transfer of wealth from creditors to debtors, so that under partial indexation borrowers have even more resources to buy capital. In fact, as will be shown in numerical exercises, under partial indexation the amplification of the monetary expansion is larger than what it would be if full indexation was possible.[20]

Part *(ii)* in Proposition 3 indicates the role of τ in the strength of the real effects following the monetary expansion. In fact, when the sterilization policy is smooth, i.e. when τ is large, borrowers further increase their capital stock in $t = 2$. This implies that they would be able to borrow more against their collateral, and their capital holdings will increase for a number of periods after the shock. This occurs because when the sterilization is smooth, then the government contracts the money supply in small amounts during several periods, and so the nominal interest rate remains below the steady state, i.e. $\hat{R}_t < 0$ for a longer time. In contrast, when the monetary expansion is reverted quickly, i.e. when $\tau \to 0$, this dynamic pattern for capital may not necessarily hold, unless further conditions are imposed.

3.2 Full indexation

The collateral constraint implies that there is an asymmetry between monetary expansions and contractions. A monetary expansion increases the price of the collateral by less than period one's inflation π_1 so that $\frac{1+\pi_1}{\beta'} \frac{B_{-1}}{p_{-1}} > q_0 K_{-1}$, so that full indexation is not feasible.[21] In contrast, a monetary contraction drives down the price of the collateral by less than period one's deflation so that $\frac{1+\pi_1}{\beta'} \frac{B_{-1}}{p_{-1}} < q_0 K_{-1}$. In this case fully-indexed contracts are feasible.

[20]The redistribution of wealth between debtors and creditors following a money shock has been emphasized by Fisher (1933).

[21]See numerical examples below.

If debt is fully indexed, linearization of equation (13) when $\pi = 0$ yields

$$\hat{K}_1 = \frac{1}{1-\beta'\rho}\left[\beta'\hat{q}_1 + \left(1-\beta'\tau - r^h\right)\varepsilon\right]. \tag{15}$$

To solve for $\hat{K}_1$ we first need to solve for $\hat{q}_1$, which in turn depends on the whole sequence $\{\hat{K}_t\}$. Solving equation (??) forward we can obtain a solution for $\hat{q}_0$ and $\hat{q}_1$, as shown in Appendix C. The expression that relates $\hat{q}_1$ with $\hat{K}_1$ is algebraically complicated.

We can use equation (15) to gain some intuition on the real effects of a monetary contraction under full indexation. Suppose initially that the real price of capital remains unchanged after the monetary shock so that $\hat{q}_1 = 0$. In this case, the real effects of the shock depend on the size of τ. Specifically, if $\tau < \overline{\tau} \equiv \frac{1}{\beta'}\left(1 - r^h\right)$, then K_1, and also Y_2, move in the same direction as the monetary shock. This is generally the case because β' is close to 1, case in which $\overline{\tau}$ is close to 1 too. Finally, this change in the distribution of capital toward the less productive agents induces an decrease in the price of capital, $\hat{q}_1 < 0$, which reinforces the initial effect of the shock. Therefore, a one-time monetary contraction under full indexation generally induces a redistribution of capital towards lenders, and decreases output. The following section illustrates the dynamics of the model with a numerical example.

4 Numerical examples

To illustrate the magnitude and persistence of monetary shocks in this economy, we assign values to the parameters of the economy and simulate the effects of a one-time 1% increase in the growth rate of money. We choose the parameters of the model to satisfy the assumptions imposed. It is worth mentioning that we were able to verify the predictions presented above for a large set of parameters. We set $\beta' = 0.995$ to simulate a time period equal to a month. Note that β' is close enough to 1, in line with many of the proofs presented above.

We normalize to unity the total stock of capital, i.e. $\bar{K} = 1$, as well as the nontradable fraction of output, i.e. $c = 1$. The production technology for lenders is: $G(K) = B(\bar{K} - K)^{\gamma}$, where

B is also normalized to unity. We set $\gamma = 0.5$ and perform sensitivity analysis. We set $n = 3$, which implies that in this economy only 25% of the agents are constrained. Finally, we choose a steady-state capital distribution of $K = 0.25$, i.e. lenders hold 25% of the total capital.

Figure 2 displays the effects of a one-time increase of 1% in the growth rate of money when $\pi = 0$ and $\tau = 0.9$. Recall that in this scenario there is partial indexation in equilibrium. The figure shows percentage deviations from steady-state values. Since τ is large, the subsequent money contraction is smooth and government bonds go back gradually to their steady-state $H^s = 0$. As is shown in the graph, this policy generates ample and persistent dampening cycles. The cycle starts with an increase in borrowers' capital holdings, as well as an increase in output. The peak of the cycle is reached about 50 months after the shock, when borrowers' capital is 30% above the steady state, while output is around 3.5% higher. It takes about 100 months before both variables reach levels below the steady state. Notice also that the minimum points reached are only 15% for capital and 2% for output, which are lower than the absolute value of the maximum points. Since the collateral constraint binds, real borrowers' debt mimics the behavior of capital.

These results emerge from the combination of two mechanisms that affect both sides of the collateral constraint: one is the asset-price effect, and the other is the interest-rate effect. First, there is an increase in real price of capital that increases the value of the collateral for a number of periods. This increase in the asset price comes from the fact that to clear the capital market, the users cost for lenders has to increase. Notice that the real price of capital is above the steady state for 50 months, which is exactly the time at which capital and bonds reach their peaks. Second, although the nominal interest rate increases in the period of the shock, it then decreases above the steady state and remains low for around 50 months. This is consistent with the behavior of the inflation rate. Further, notice that the nominal rate is below the steady state for a number of periods, which indicates that this model can generate a persistent liquidity effect following a money expansion.

Figure 3 replicates the same experiment of Figure 2, but with $\tau = 0.1$. As observed in the figure, in this case the policy reversal after the shock is faster. Otherwise, the results are similar to the ones in Figure 2. One main difference though is that with a low τ, the multiplier for capital and output during the first months is much larger than when τ is high. This large multiplier is associated with the larger decrease in inflation in period $t = 2$, as observed in the graph. In fact, in the experiment presented here this multiplier almost doubles when going from $\tau = 0.9$ to $\tau = 0.1$. In particular, $\hat{P}_2 = 0.27\%$ when $\tau = 0.1$, while it is 0.15% when $\tau = 0.9$. However, this gap closes after some months. In fact when $\tau = 0.1$ the peak occurs at $\hat{P}_{50} = 3.5\%$, while it is 3.53% when $\tau = 0.9$.

To illustrate the case when full indexation is effective, Figure 4 displays the effects of a one-time decrease of 1% in the growth rate of money when $\pi = 0$ and $\tau = 0.9$. Recall that full indexation with no renegotiation can only occur when the borrower does not have an incentive to repudiate the debt contract. In this model, this occurs when there is a money contraction. In this case, the model still exhibits persistence, but the amplitude of the effects is much smaller than the ones observed in Figures 1 and 2. In fact, output reaches a trough of only about -0.1%.[22]

Finally, to illustrate the case when $\pi \neq 0$, Figure 5 displays the effects of a one-time increase of 1% in the growth rate of money when $\pi = 0.4\%$ per month, which corresponds to a steady-state annual inflation rate of about 5%. In this experiment, the government chooses a monetary contraction in period $T = 5$ that allows for the transversality condition of government debt to hold.[23] Results are similar to those of Figure 2, except that now the peak in output is of a magnitude of 2% above the steady state. In this case, to get the same output peak of Figure 2 we would need a one-time money expansion of 1.6%. In general, when $\pi > 0$ we still obtain highly persistent effects, but the magnitude varies somewhat depending on the government's choice of T.[24]

Kochelakota (2000) concludes that the magnitude of the amplification of shocks in economies

[22]In this model expansions are larger than contractions. This asymmetry is the opposite to that found by Kocherlakota (2000). The reason for the difference is that here collateral constraints are always binding for borrowers, while this is not the case in Kocherlakota (2000). Evidence on the U.S. unemployment rate shows downward movements that are sharper and quicker than upward movements. However, as shown by Falk (1986) this evidence is not compelling for real gross national product, investment and productivity in the U.S., as well as for industrial production in a sample of other countries.

[23]See details on this in Appendix D.

[24]When $\pi \neq 0$, if $\hat{g}_1 = 0$, it is possible that $\hat{R}_1 > 0$. This is so because as explained in the text, one important condition to obtain $\hat{R}_1 < 0$ following a one-time monetary contraction is that $\hat{v}_2 > 0$. Since $\hat{g}_1 > 0$ contributes to have $\hat{v}_2 > 0$, then it may be possible that with $\hat{g}_1 = 0$ the increase in v_2 is not enough to achieve $\hat{R}_1 < 0$.

with credit constraints is particularly sensitive to the value of the factor shares in the production function. Although γ is not the capital share in our model because here we have agents with heterogeneous production functions, we perform sensitive analysis for this parameter. In particular, we find that for the experiment in Figure 2, as γ increases, the amplitude of the expansion also increases. For instance, if $\gamma = 0.8$, the peak observed in output is around 6.6%. Also, if $\gamma = 0.1$, then output increases up to 1.6%, which is still larger than the size of the shock (1% increase in money supply). Thus, although we also find that amplification varies with γ, future work would need to involve careful calibration of the model economy. Our only purpose here is to illustrate the dynamics generated by our model following a one-time money shock.

5 Concluding comments

This paper presents a novel approach to the propagation of monetary shocks by combining collateral and cash-in-advance constraints, in a world where changes in money supply occur via open-market operations. We find that a one-time exogenous monetary shock generates persistent movements in aggregate output, whose amplitude depends on the degree of debt indexation. Monetary expansions can trigger a large upward movement in output, while monetary contractions give rise to a smaller downward movement. This asymmetry occurs because full indexation of debt contracts can only be effective following a monetary contraction. In contrast, following a monetary expansion indexation can only be *partial* because debtors end up paying back just the market value of the collateral. Due to the existence of both cash-in-advance and collateral constraints, monetary shocks trigger a highly persistent dampening *cycle* rather than a smoothly declining deviation.

One of the limitations of the model presented here is that since the capital stock is fixed, we can only observe redistribution of capital, but not comovements in output and investment. Also, the model we used here is simple and stylized, and some of the functional forms are not general. For instance, utility functions for borrowers and lenders are linear. Even though Kiyotaki (1998) has shown that persistence and amplification of shocks in the real-economy model of Kiyotaki and Moore (1997) would still hold with concave utility, in future work we plan to explore the implications of changes in functional forms in our model. Finally, it would be interesting, once we modify some of the functional forms, to perform a more careful calibration. We plan to modify the model to resolve some of these limitations in future research. In spite of these limitations, our model is simple enough to provide insights on how credit-market imperfections work in a monetary economy where the asset that serves as a collateral is also a factor of production.

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A Proof of optimal solution for borrowers

We need to prove the claim that borrowers' optimal plan is to consume only the nontradable fraction of output, i.e. $x_t = cK_{t-1}$, to borrow up to the limit and to invest all remaining resources. To do that we compare the utility achieved under the different alternative plans. The first one is to follow the proposed *investment path*. Alternatively, borrowers can consume or save. For these last two alternatives, we only consider single deviations from the investment path at date $t = 0$.[25]

Consider the borrower's marginal utility of investing p_0 dollars given that all aggregate variables remain unchanged at their steady state levels. For simplicity let $\pi = 0$. In steady state, we have $R = 1/\beta'$and $q = a/(1-\beta')$. Therefore, for given prices and aggregate variables at their steady state levels, equations (1), (2) and (3) can be rewritten as:

$$qk_t + (c-q)k_{t-1} = \frac{m_{t-1}^d}{p_{t-1}} \tag{A1}$$

$$\frac{b_t}{p_t} = q\beta' k_t \tag{A2}$$

$$\frac{m_t^d}{p_t} = (a+c)k_{t-1} + \frac{b_t}{p_t} - R\frac{b_{t-1}}{p_{t-1}}. \tag{A3}$$

Replacing the borrowing constraint into the budget constraint,

$$\frac{m_t^d}{p_t} = (a+c-q)k_{t-1} + q\beta' k_t. \tag{A4}$$

Substituting (A4) into (A1) and solving for k_t:

$$k_t = \left(\beta' + 1 - \frac{c}{q}\right)k_{t-1} + \left(\frac{a+c}{q} - 1\right)k_{t-2}. \tag{A5}$$

¿From the steady state value of q we have that $\left(\beta' + 1 - \frac{c}{q}\right) = 2 - (1-\beta') - \frac{c}{q} = 2 - \frac{a+c}{q}$ $= 2 - r^h$. Let $r^h \equiv \frac{a+c}{q}$. We can rewrite (A5) as:

$$k_t = \left(2 - r^h\right)k_{t-1} + (r^h - 1)k_{t-2} \tag{A6}$$

It is easy to check that the roots of the associated characteristic polynomial are 1 and $1 - r^h$. Therefore, k_t can be expressed as:

$$k_t = A_1 + A_2(1-r^h)^t. \tag{A7}$$

where constants A_1 and A_1 need to be determined. Under the proposed guess, the optimal strategy for borrowers is to use the extra p_0 dollars to invest in capital. With this amount, the borrower

[25] Following the logic of Kiyotaki and Moore (1997), "we appeal to the principle of unimprovability", which states that to prove that our proposed strategy of investing all the extra p_0 dollars is optimal, we need to consider only single deviations from this plan at date $t = 0$.

can buy $k_0 = 1/q$ units of capital at $t = 0$. This allows him to borrow $q\beta' k_0 = \beta'$ additional units of output.[26] At $t = 1$, consumption increases by ck_0 units so that from the additional resources, $\beta' - c/q$ can be used to buy capital. Therefore, investment is given by: $k_1 - k_0 = \frac{\beta' - c/q}{q}$, so that $k_1 = k_0(\beta' - c/q + 1)$. With these two initial values (k_0,k_1), constants $A_1$and A_2 can be determined as follows:

$$A_2 = \frac{k_0 - k_1}{r^h} \text{ and } A_1 = \frac{1}{r^h}\left[k_1 - (1 - r^h)k_0\right].$$

Utility under the investment path is given by:

$$\begin{aligned} U^{inv} &= \beta c \sum_{t=0}^{\infty} \beta^t k_t \\ &= \beta c \sum_{t=0}^{\infty} \beta^t \left[A_1 + A_2(1 - r^h)^t\right] = cA_1 \frac{\beta}{1-\beta} + cA_2 \frac{\beta}{1 - \beta(1 - r^h)} \\ &= \frac{c}{q}\frac{\beta}{1-\beta}\frac{1}{1-\beta(1-r^h)}. \end{aligned}$$

To show that higher utility is attained in the investment path than in the *consumption path*, we need to find conditions under which:

$$\frac{c}{q}\frac{\beta}{1-\beta}\frac{1}{1-\beta(1-r^h)} > 1$$

We can transform the expression above to obtain:

$$\frac{c}{a} > \frac{(1-\beta)}{\beta}\frac{(1-\beta)}{(1-\beta')} + (1-\beta)\frac{a+c}{a}.$$

Since $\frac{(1-\beta)}{\beta} < \frac{(1-\beta)}{\beta^2}$ then a sufficient condition for the utility from the investment path being higher is:

$$\frac{c}{a} > \frac{(1-\beta)}{\beta^2}\left(\frac{(1-\beta)}{(1-\beta')} + 1\right)$$

which corresponds to Assumption 2 in the text.

To complete the proof we need to show that higher utility is attained in the investment path than in the *saving path*. Borrowers can save the p_0 dollars and use the return R to commence a strategy of maximum levered investment from date $t = 1$ onwards. Then, all we need to show is that the returns from saving p_0 dollars in period $t = 0$ are lower than the return from investing at $t = 0$. Since from Assumption 1, $\beta' > \beta$, using Assumption 2 is easy to show that $\beta' > \frac{a}{a+c}$. Thus,

[26]Note that p_0 dollars are equivalent to one unit of output at $t = 0$ prices. Also, by borrowing extra $b_0 = \beta'$, the agent can demand extra β' real money balances in the third subperiod of $t = 0$, in order to buy additional capital in the first subperiod of $t = 1$.

$$
\begin{aligned}
1+r^h &= 1+\frac{a+c}{q}=1+\frac{(a+c)(1-\beta')}{a}>1+\frac{1-\beta'}{\beta'} \\
&= \frac{1}{\beta'}=R.
\end{aligned}
$$

Therefore, $1+r^h > R$, which guarantees that the investment path yields more utility than the alternative savings path. This completes the proof that the proposed solution is an equilibrium. We have presented an analytical proof for $\pi = 0$. For $\pi \neq 0$ it is not possible to provide an analytical proof. However, for all the numerical simulations in the text, we have verified in the computer that the decision rules for the borrower are optimal.

B Proof of Proposition 2

Let $\pi = 0$ so that in steady state $u = a$. Equation (11) in the text reads:

$$
\theta_0 \hat{k}_t = \theta_1 \hat{k}_{t-1} + \theta_2 \hat{k}_{t-2} + \mu_0 \tau^{t-2} \hat{y}_0 \tag{B1}
$$

where:

$$
\theta_0 = 1 + (1-2\beta')\rho
$$

$$
\theta_1 = (1-r^h)(1-\rho) + 1 + (1-2\beta')\rho - (1-\beta')\frac{1}{\eta}
$$

$$
\theta_2 = -(1-r^h)(1-\rho)
$$

$$
\mu_0 = -(R-\tau)\left[\left(1-2\beta'\right)\tau + (1-r^h)\right]
$$

Since the particular solution for the equation above is

$$
\hat{k}_p = \frac{\mu_0 \hat{y}_0 \tau^t}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}
$$

then the general solution is given by:

$$
\hat{k}_t = A_1 \lambda_1^t + A_2 \lambda_2^t + A_\tau \tau^t \hat{y}_0 \tag{B2}
$$

where $A_\tau = \frac{\mu_0}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}$ is a constant and the eigenvalues λ_1 and λ_2 satisfy: $\lambda_1 \lambda_2 = \frac{-\theta_2}{\theta_0}$ and $\lambda_1 + \lambda_2 = \frac{\theta_1}{\theta_0}$. Finally, the solutions for constants A_1 and A_2 can be obtained from: $\hat{k}_1 = A_1 \lambda_1 + A_2 \lambda_2 + A_\tau \tau \hat{y}_0$ and $\hat{k}_0 = A_1 + A_2 + A_\tau \hat{y}_0$.

B.1 Cycles

The dynamic properties of equation (B1) depend on the eigenvalues associated to the homogeneous difference equation $\theta_0 \hat{K}_t = \theta_1 \hat{K}_{t-1} + \theta_2 \hat{K}_{t-2}$ which are given by:

$$\lambda_1, \lambda_2 = \frac{\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_0\theta_2}}{2\theta_0}.$$

The necessary and sufficient condition for cycles is $\theta_1^2 + 4\theta_0\theta_2 < 0$. Note that $\theta_1$can be rewritten as:

$$\theta_1 = \theta_0 - \theta_2 - (1-\beta')\frac{1}{\eta}. \tag{B3}$$

Adding and subtracting proper terms, θ_2 can be rewritten as

$$\theta_2 = \xi(\beta') - \theta_0 \tag{B4}$$

where:

$$\begin{aligned} \xi(\beta') &\equiv 2\rho(1-\beta') + r^h(1-\rho) \\ &= (1-\beta')(2\rho + \frac{a+c}{a}(1-\rho)) > 0. \end{aligned}$$

From (B3) and (B4), θ_1 can be written as:

$$\theta_1 = 2\theta_0 - \zeta(\beta') \tag{B5}$$

where:

$$\begin{aligned} \zeta(\beta') &\equiv (1-\beta')(2\rho + \frac{a+c}{a}(1-\rho) + \frac{1}{\eta}) \\ &< (1-\beta')\left[\max\left\{2, \frac{a+c}{a}\right\} + \frac{1}{\eta}\right]. \end{aligned}$$

Finally, from (B3) and (B4), $\lim_{\beta' \to 1} \theta_2 = \theta_0$ and $\lim_{\beta' \to 1} \theta_1 = 2\theta_0$.

B.2 Proof of Proposition

To show that for β' sufficiently large the model exhibits cycles, it needs to be proven that $\theta_1^2 + 4\theta_0\theta_2 < 0$. Use (B4) and (B5) to get:

$$\begin{aligned} \theta_1^2 + 4\theta_0\theta_2 &= (2\theta_0 - \zeta(\beta'))^2 - 4\left[\theta_0 - \xi(\beta')\right]\theta_0 \\ &< -4\theta_0(1-\beta')\frac{1}{\eta} + (1-\beta')^2\left[\max\left\{2, \frac{a+c}{a}\right\} + \frac{1}{\eta}\right]^2. \end{aligned}$$

Note that $\lim_{\beta' \to 1} \frac{1}{\eta} = -\frac{G''((\bar{K}-K^1)/n)K^1}{nG'((K-K^1)/n)} > 0$ where K^1 is the solution of (10) for β' equal to 1 and

$\pi = 0$. Therefore, the second term in the last expression approaches to zero faster than the first term as $\beta' \to 1$. Note that $\theta_0 \frac{1}{\eta}$ remains bounded above since θ_0 approaches $1 - \alpha^N(K^1) > 0$ and the fact that $\frac{1}{\eta}$ approaches a constant greater than zero. Thus, for β' large enough the first term dominates and the expression is negative. ■

It is also useful to state solution (B2) in its polar representation (See Allen, 1959, page 189)

$$\hat{K}_t = Ar^t \cos(\omega t + \phi) + A_\tau \tau^t,$$

where A and ϕ are constants that can be determined from the initial conditions, and

$$r = \sqrt{-\theta_2/\theta_0},$$
$$\omega = \cos^{-1}\left(\frac{\theta_1/\theta_0}{2r}\right).$$

Stability is guaranteed if the modulo r is less than 1, a result that follows from (B5) for large β'. In addition, r is close to 1 when β' is close to 1. Thus, the difference equation displays persistent dampening cycles.

C Forward looking solution for asset prices

This appendix gives the solution for $\hat{q}_0$ and $\pi = 0$. From equation (??) in the text:

$$\hat{q}_t - \beta' \hat{q}_{t+1} = (1 - \beta')\left(\frac{1}{\eta} - \rho\right)\hat{K}_t + (1 - \beta')\rho \hat{K}_{t+1} - (1 - \beta')\hat{g}_{t+1}$$

iterate forward and use the transversality condition $\hat{q}_\infty = 0$ to rule out bubbles in the price of capital to obtain:

$$\begin{aligned}\frac{\hat{q}_0}{(1-\beta')} &= \sum_{j=0}^{\infty} \beta'^j \left[\left(\frac{1}{\eta} - \rho\right)\hat{K}_j + \rho \hat{K}_{j+1} - \hat{g}_{j+1}\right] \\ &= \left(\frac{1}{\eta} - \rho\right)\hat{K}_0 + \sum_{j=0}^{\infty} \beta'^j \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\hat{K}_{j+1} - \sum_{j=0}^{\infty} \beta'^j \hat{g}_{j+1}.\end{aligned}$$

Using the solution for the non-homogeneous second order difference equation (B1) we have:

$$\begin{aligned}\frac{\hat{q}_0}{(1-\beta')} &= \left(\frac{1}{\eta} - \rho\right)\hat{K}_0 + \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\sum_{j=0}^{\infty} \beta'^j \left(A_1\lambda_1^{j+1} + A_2\lambda_2^{j+1} + A_\tau \tau^{j+1}\hat{g}_0\right) \\ &\quad + (R - \tau)\hat{g}_0 \sum_{j=0}^{\infty} \beta'^j \tau^j\end{aligned}$$

which after some algebra yields:

$$\frac{\widehat{\varphi}_0}{(1-\beta')} = \left(\frac{1}{\eta}-\rho\right)\widehat{R}_0 + \left(\beta'\left(\frac{1}{\eta}-\rho\right)+\rho\right)\frac{(\lambda_1 A_1+\lambda_2 A_2)-\beta'\lambda_1\lambda_2\left(A_1+A_2\right)}{1-\beta'(\lambda_1+\lambda_2)+\beta'^2\lambda_2\lambda_2}$$
$$+\left(\beta'\left(\frac{1}{\eta}-\rho\right)+\rho\right)\frac{\tau A_\tau \widehat{g}_0}{1-\beta'\tau}+\frac{(R-\tau)\widehat{g}_0}{1-\beta'\tau}.$$

Finally using $\widehat{R}_0 = 0$ and the properties of λ_1, λ_2 and $\widehat{R}_1$ from Appendix B we get, after some algebra:

$$\frac{\widehat{\varphi}_0}{(1-\beta')} = \frac{\left(\beta'\left(\frac{1}{\eta}-\rho\right)+\rho\right)\theta_0}{\theta_0-\beta'\theta_1-\beta'^2\theta_2}\widehat{R}_1+\frac{(R-\tau)\widehat{g}_0}{1-\beta'\tau}$$
$$+\left(\beta'\left(\frac{1}{\eta}-\rho\right)+\rho\right)\left(\frac{\tau}{1-\beta'\tau}-\frac{(\theta_0\tau+\beta'\theta_2)}{\theta_0-\beta'\theta_1-\beta'^2\theta_2}\right)A_\tau\widehat{g}_0$$

which solves for $\widehat{\varphi}_0$ as a function of $\widehat{R}_1$. Also, the following equation relates $\widehat{\varphi}_0$, $\widehat{\varphi}_1$ and $\widehat{R}_1$:

$$\widehat{\varphi}_0 = \beta'\widehat{\varphi}_1 + \left(1-\beta'\right)\rho^\pi\widehat{R}_1 + (1-\beta')(R-\tau)\widehat{g}_0$$

D Solution for $\pi > 0$

When the steady-state inflation is not zero, but $\pi > 0$, then the simple rule that following a one-time money shock at $t = 0$ we can guarantee convergence of d_t back to the steady state by imposing $\tau < 1$ does not hold anymore. Recall that since $H_t^s = \tau H_{t-1}^s$ and when $\pi = 0$ we have $H^s = 0$, then $\tau < 1$ is enough to guarantee that H_t^s eventually converges to zero. In contrast, this is not the case when $\pi > 0$ then $d > 0$. Thus, when $\pi > 0$ the "sterilization" policy needs to be changed.

In particular, assume that the economy starts off the steady state and at time $t = 0$ there is an unexpected one-time increase in growth rate of money $\varepsilon > 0$, i.e. $\widehat{g}_0 = \frac{\varepsilon}{1+\pi}$. In this case, the government chooses a period $t = T$ such that from T on, the growth rate of money supply is zero, i.e. $\widehat{g}_t = 0$ for $t > T$. What this implies is that for $t > T$, the law of motion of $\widehat{d}_t$ is given by:[27]

$$\widehat{d}_t = \frac{1}{\beta'}\widehat{R}_t + \frac{1}{\beta'}\widehat{d}_{t-1}$$

which is clearly unstable, since $\beta' < 1$. Iterating forward on the equation above and imposing the transversality condition that $\widehat{d}_\infty = 0$, we obtain that $\widehat{d}_{T-1}$ must satisfy:

$$\widehat{d}_{T-1} = -\sum_{\tau=0}^{\infty}\beta'^\tau\widehat{R}_{\tau+T}$$

to guarantee convergence back to the steady-state. Further, since using the law of motion of money

[27] This equation is the linearized version of the law of motion of the money supply when $\widehat{g}_t = 0$.

supply we have that $\hat{g}_{T-1}$ is given by:

$$\hat{g}_{T-1} = -\frac{d}{1+d}\hat{d}_{T-1} + \frac{d}{\beta'(1+d)}\hat{K}_{T-1} + \frac{d}{\beta'(1+d)}\hat{d}_{T-2}$$

so that $\hat{g}_{T-1}$ depends on $\hat{d}_{T-1}$. In summary, when the government chooses a period T such that $\hat{g}_T = 0$, it must also choose $\hat{g}_{T-1}$ to satisfy the transversality condition. Further for periods $1 \leq t < T-2$ we allow the government to choose any exogenous law of motion for $\hat{g}_t \leq 0$, i.e. any rule in which the monetary expansion at time $t = 0$ is reverted. For instance, a natural choice would be a gradual money contraction up to period $T-2$ and a choice of $\hat{g}_{T-1}$ that satisfies the condition above.

When $\pi > 0$, the dynamics of capital are described by:

$$\theta_0^\pi \hat{K}_t = \theta_1^\pi \hat{K}_{t-1} + \theta_2^\pi \hat{K}_{t-2} + \frac{1}{1+\pi}\left[\left(1-2\beta'\right)\hat{g}_t + (1-r^h)\hat{g}_{t-1}\right]$$

where:

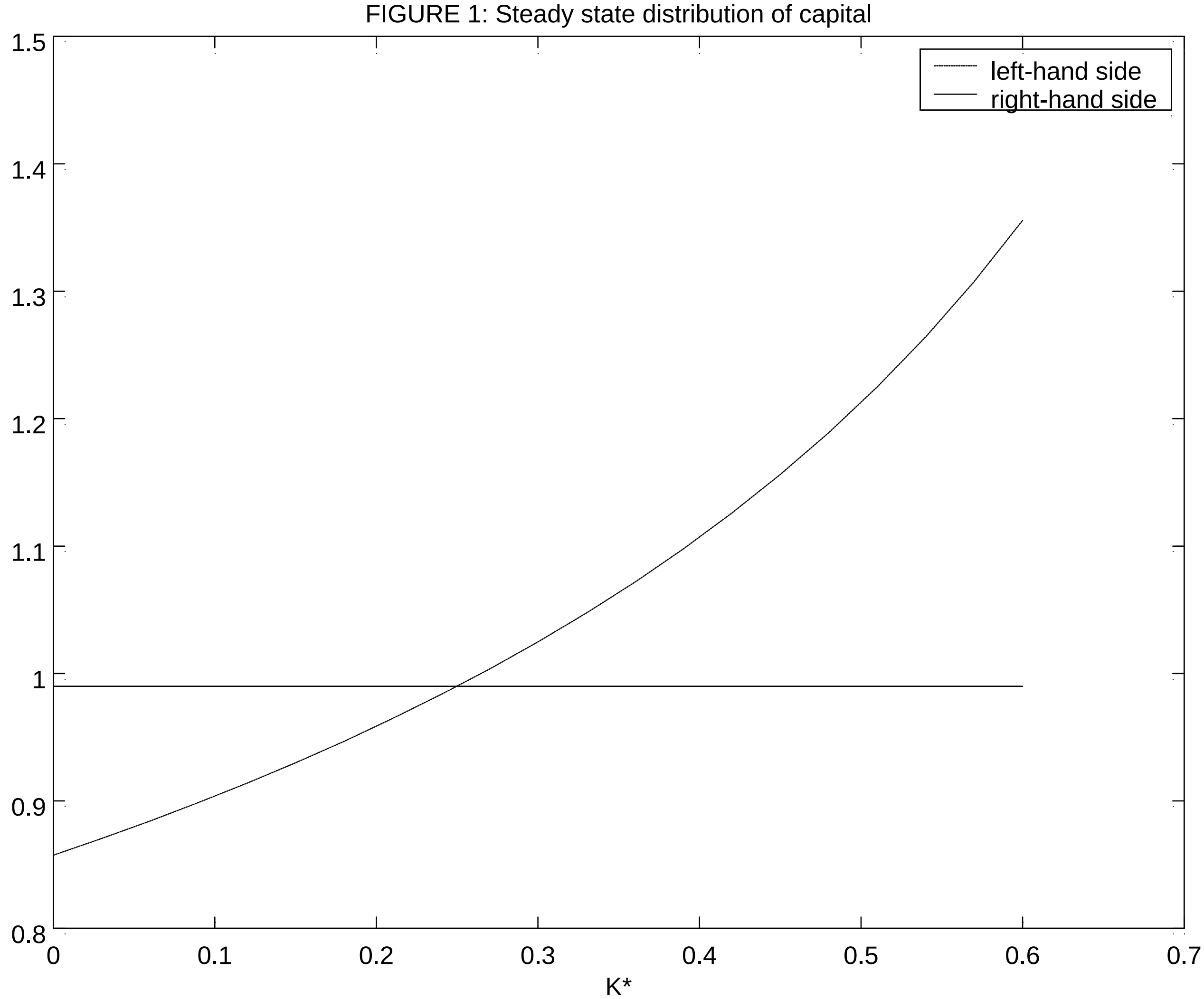
$$\theta_0^\pi = 1 + (1-2\beta')\frac{\rho}{(1+\pi)}$$

$$\theta_1^\pi = \frac{(1-r^h)(1-\rho)}{(1+\pi)} + 1 + (1-2\beta')\frac{\rho}{(1+\pi)} - \frac{(1-\beta')}{\eta(1+\pi)}$$

$$\theta_2^\pi = -\frac{(1-r^h)(1-\rho)}{(1+\pi)}.$$

Using the dynamic equation of capital, as well as the transversality condition for government debt, the law of motion of money supply and the forward-looking solution for capital prices it is possible to construct a system of 5 equations in 5 unknowns: $\hat{K}_{T-1}$, $\hat{q}_{T-2}$, $\hat{q}_{T-1}$, $\hat{d}_{T-1}$ and $\hat{g}_{T-1}$. Since this system is a function of past values $\hat{K}_{T-3}$, $\hat{K}_{T-2}$ and $\hat{d}_{T-2}$ an iterative procedure that starts with a guess for $\hat{K}_1$ must be implemented to find the solution. Details on the solution procedure are available from the authors upon request.

FIGURE 1: Steady state distribution of capital



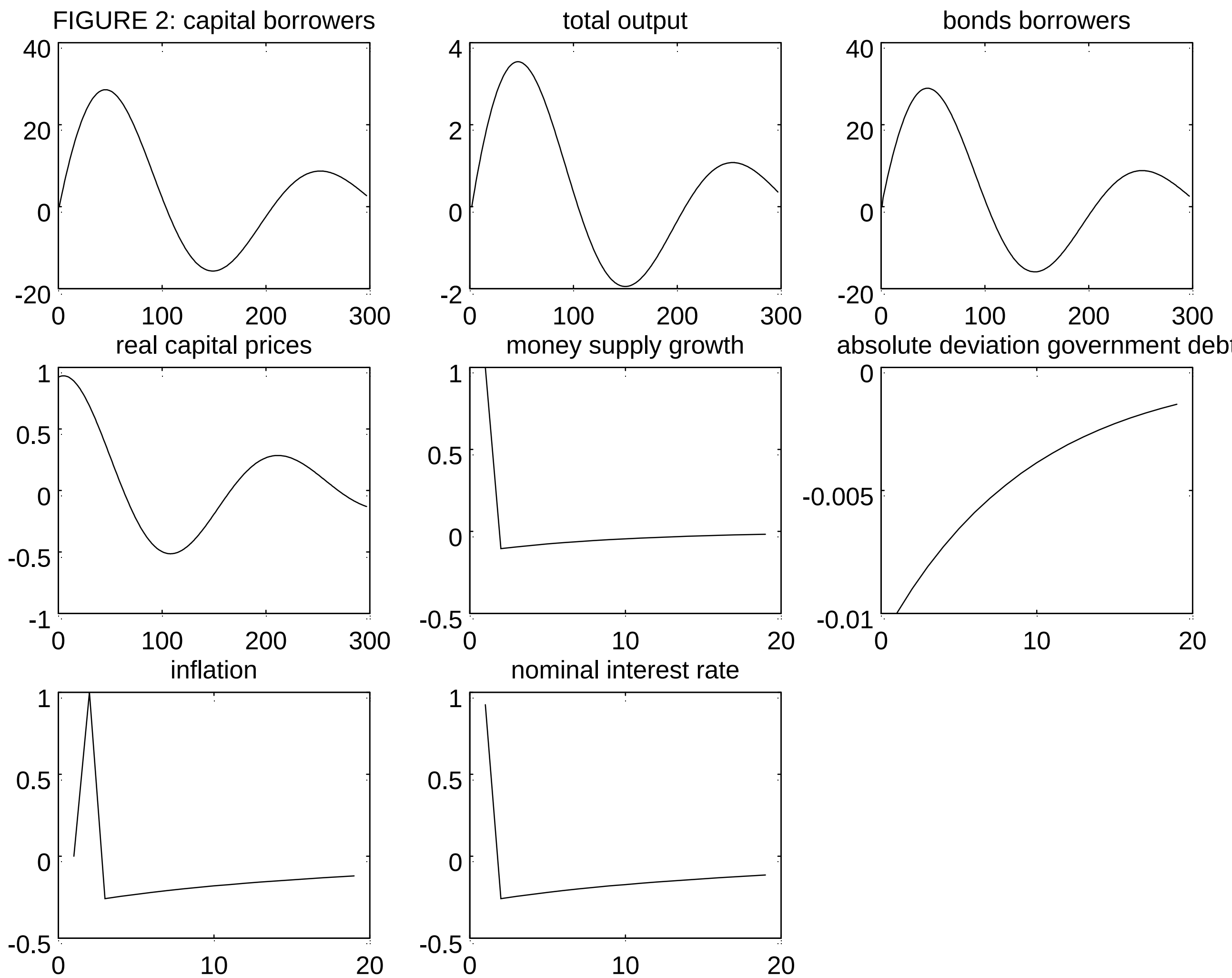


FIGURE 2: capital borrowers
total output
bonds borrowers
real capital prices
money supply growth
absolute deviation government debt
inflation
nominal interest rate
40
20
0
-20
4
2
0
-2
0
100
200
300
1
0.5
0
-0.5
-1
-0.005
-0.01
10
20

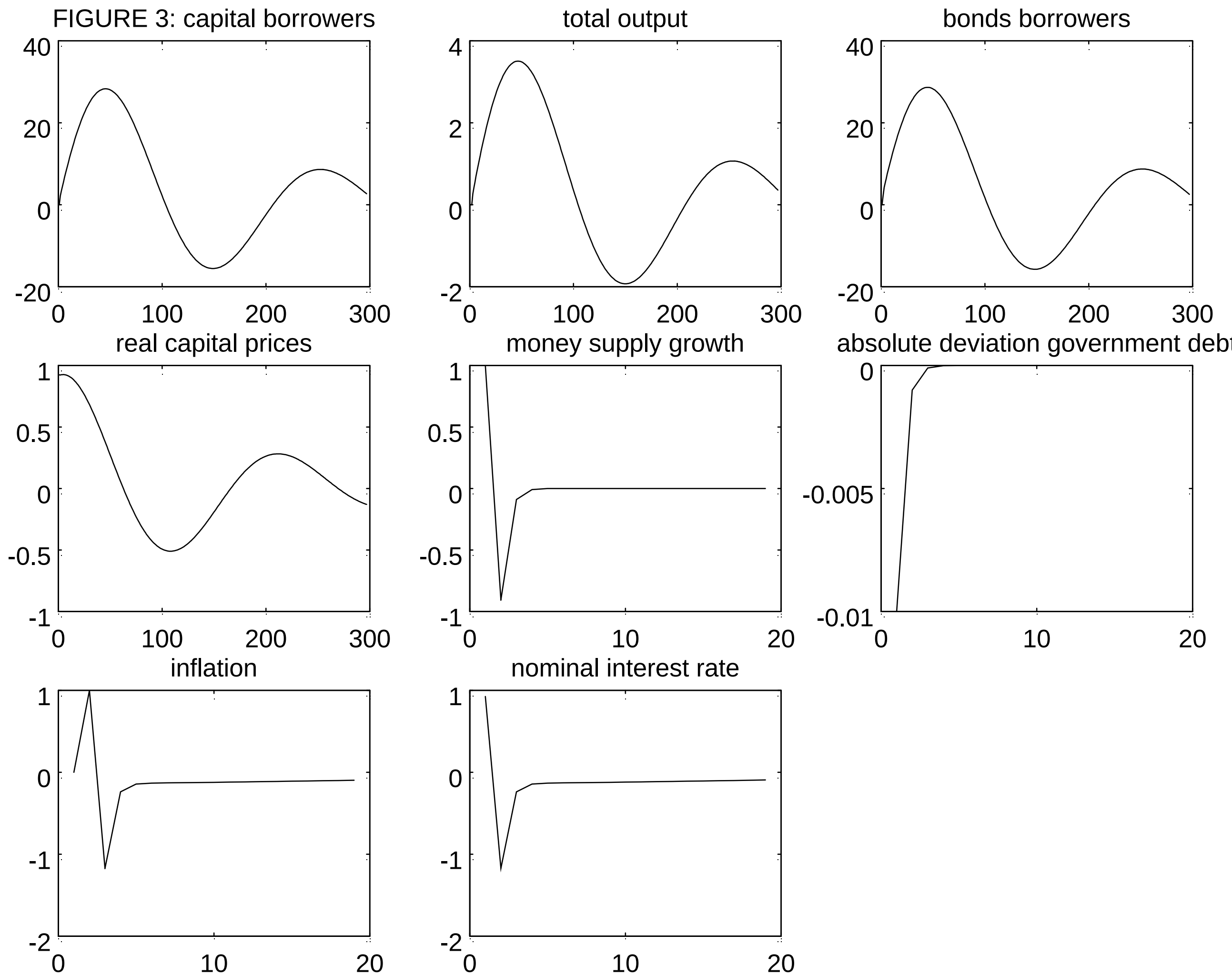


FIGURE 3: capital borrowers
40
20
0
-20
0
100
200
300
total output
4
2
0
-2
0
100
200
300
bonds borrowers
40
20
0
-20
0
100
200
300
real capital prices
1
0.5
0
-0.5
-1
0
100
200
300
money supply growth
1
0.5
0
-0.5
-1
0
10
20
absolute deviation government debt
0
-0.005
-0.01
0
10
20
inflation
1
0
-1
-2
0
10
20
nominal interest rate
1
0
-1
-2
0
10
20

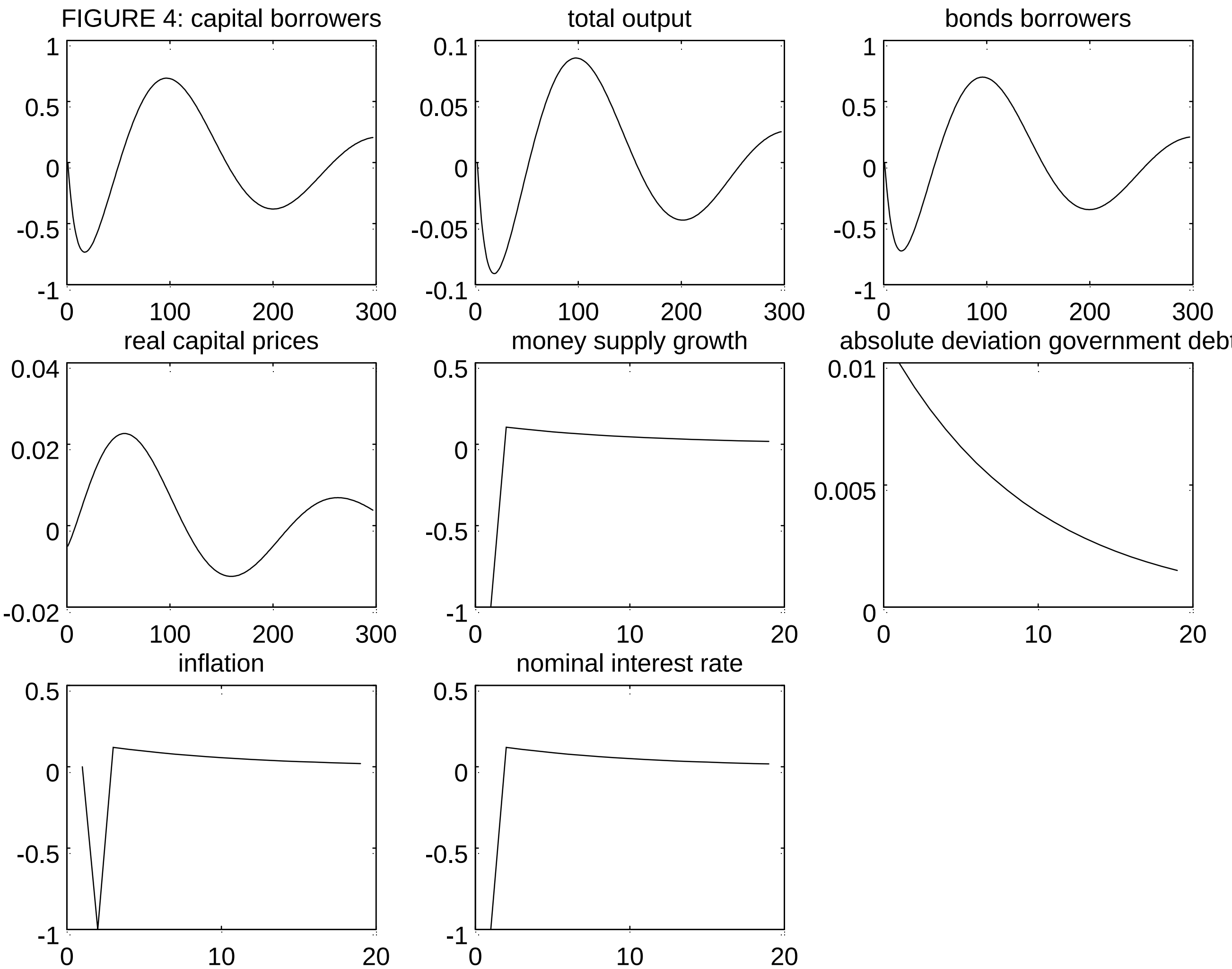


FIGURE 4: capital borrowers
total output
bonds borrowers
real capital prices
money supply growth
absolute deviation government debt
inflation
nominal interest rate

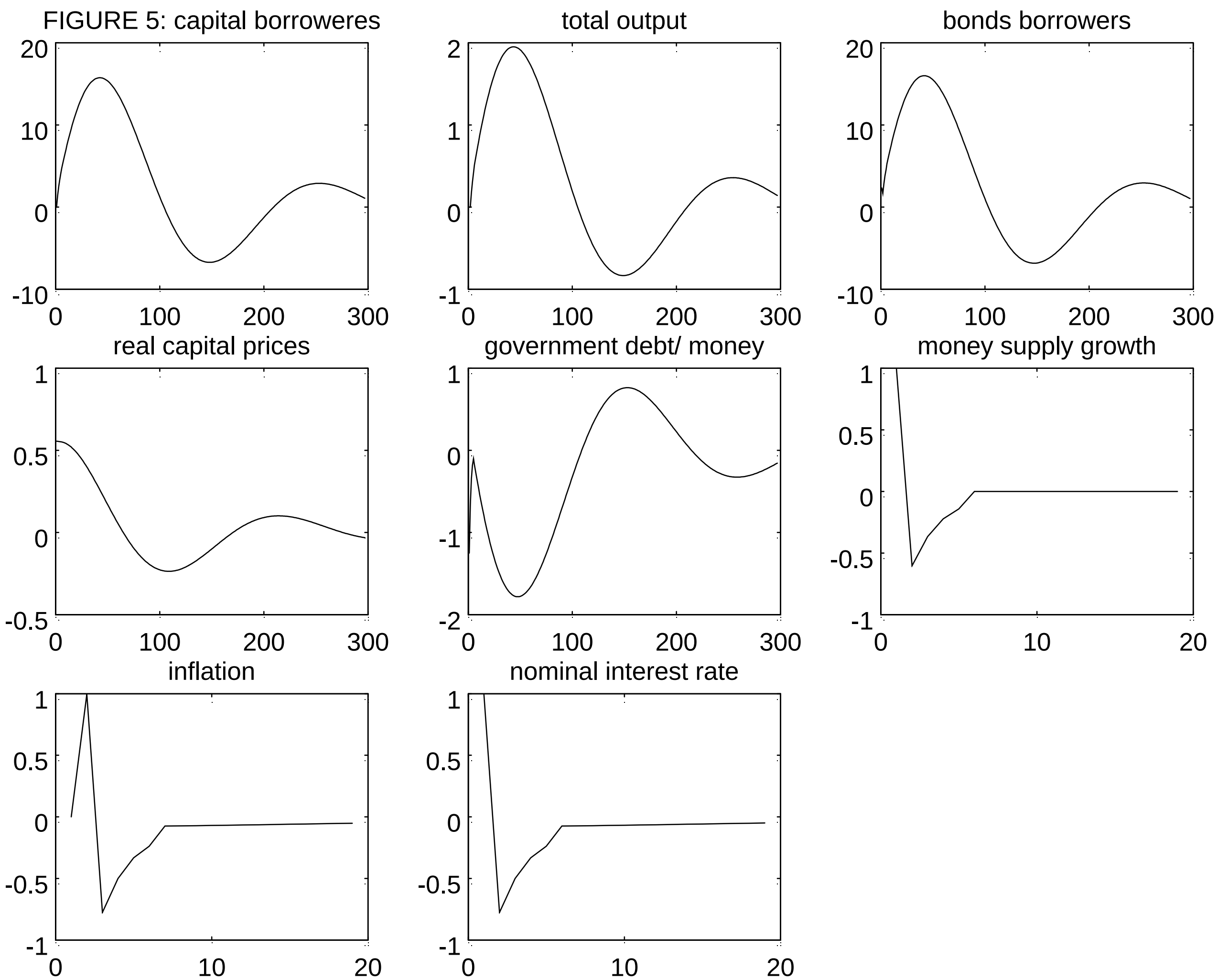


FIGURE 5: capital borroweres
total output
bonds borrowers
real capital prices
government debt/ money
money supply growth
inflation
nominal interest rate