Social Norms and Choice: A Weak Folk Theorem for Repeated Matching Games.

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Abstract

A folk theorem for repeated matching games is established that holds if the stage game is not a pure coordination game. It holds independent of population size and for all matching rules—including rules that depend on players choices or the history of play. This paper also establishes an equilibrium condition and using this discovers two differences between the equilibria of repeated matching games and standard repeated games. Trigger strategies are not equilibria and there is no simple optimal penal code.

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1 Introduction

It is well established that a small group—which is committed to interacting in the future—can overcome its members' incentives to act selfishly and cooperate. However, situations with this type of commitment are rare in economics. For example, consumers usually choose a supplier each time they buy a good. Can the consumers' and suppliers' short term incentives to not pay their bills and provide low quality be overcome? This paper shows they can be if people know each other's reputation.

The model used in this analysis is a repeated matching game. This paper is the first to extensively compare this game with the more familiar standard repeated game. After developing the equilibrium condition, two key differences between the set of equilibria in standard repeated games and repeated matching games are illustrated. In this model simple optimal penal codes (Abreu [1]) and trigger strategies are not equilibria.

This research builds on the work of Okuno-Fujiwara and Postlewaite [[18] OFP hereafter] and Kandori [15] in repeated matching games. OFP formalized the strategies used in this literature and showed that cooperation could be achieved with a continuum of buyers and sellers. Kandori extended these results to finite populations matched randomly. Here I generalize Kandori [15] to games where players choose whom to be matched with and to interactions with more than two participants.

As an example of the general model consider a day labor market. Every morning employers show up, and select several people from the laborers present to work for the day. The results here allow the employers to choose who they want to work for them—as long as they choose fewer than some upper bound. Two restrictions are that there is no excess supply of jobs or workers and all employers and employees are nearly the same.

In Kandori [15] random matching was required but it is hard to imagine a realistic example of either market where all matching is random. As will be shown if even one player can choose who to interact with then the strategies used in Kandori [15] fail. This might seem odd since players are basically the same, but after people deviate players are different. Someone who just cheated will have to be punished and employers may prefer to hire someone else.

The equilibria will be *social norms* (OFP [18]). These strategies are based on players' *social status* (or reputation). Given the social status of the people interacting a *social standard of behavior* tells them what action to take today, and then the *transition rule* updates the social status of players depending on what she and the people she is interacting with have done in the past. These strategies will have to be sequential equilibria, and they also satisfy several other restrictions.

These further restrictions are motivated by the fact that the equilibria only use *local* *information processing* (OFP [18]). A strategy uses *local information processing* if it only uses information that the players involved in an interaction should know. Social norms only use what a player and the people she interacts with have done in the past, and thus satisfy this restriction. Without this restriction a strategy might require a great deal of knowledge on the part of players, an amount that would easily be untenable in a large society. Due to this restriction Hasker [14] shows that cooperation can be sustained by players sending messages back and forth, without any other source of information.

Given this restriction, should something be considered an equilibrium if too much information causes it to break down? If information sources such as newspapers have to be ruled out to make a strategy an equilibrium the equilibrium is not interesting. Local information processing was required to be certain the strategy didn't require too much information, *straightforward* (Kandori [15]) is required to make certain that the strategy doesn't require that players have too little information. It makes certain that there is no value to this extra information by requiring that players would do the same thing even if they had full information.

The motivation for the final restriction on the set of equilibria is based on the same principal. Surely we don't want to require our players to know the way everyone decides who to interact with? And what is the difference between requiring them to interact with one group of people forever and always using a certain matching rule? Clearly the difference is philosophical, in both cases an untenable amount of commitment is required. Thus it seems an equilibrium of a repeated matching game should not depend on a particular matching rule, and *universal* equilibria satisfy this restriction (Kandori [15], with new terminology.) This requirement also includes that the strategy must work for all population sizes, obviously desirable and relatively trivial if the strategy workds for all matching rules.

The effect of these refinements is to make social norms extremely simple strategies to follow. Given these restrictions a player can have *any* beliefs and know that the best thing for them to do is just follow the social standard of behavior. As long as all players act in their best interests, a given player does not need to know anything but the social status of the people she is interacting with today. All the rest will take care of itself.

The end result is a folk theorem that holds in any stage game that is not one of pure cooperation—the most general sufficient condition for folk theorems in the standard repeated game (Abreu, Dutta and Smith [2]). The results in Abreu et al. are more general because the minmax can be in mixed strategies, but this is not a great weakness in matching games where the option of not interacting is both in pure strategies and the minmax. It is also shown here that the folk theorem holds without correlated actions, and holds if payoffs are slightly heterogenous.

The literature on repeated matching games was started by Rosenthal [20] and Rosenthal and Landau [21]. OFP [18] define local information processing and show that with a continuum of players a trigger strategy can establish a folk theorem. Kandori [15] presents a folk theorem for many two player random matching games; defines the straightforward refinement, and is the first to require universality.

Kandori [[15] Sect. 4] and Ellison [7] analyze what can be done with less information. In games with a dominant strategy and uniform matching these papers find that a "contagion" strategy sometimes works. The initial defection leads other players to defect, until everyone is playing the dominant strategy all of the time. This strategy works only if the population is small relative to players patience and requires that players have "too little" information.

Ahn and Suominen [3] and Hasker [14] have considered what happens if all information is generated through communication. Ahn and Suominen assume players talk to their neighbors (word of mouth). Those results are similar to Kandori [[15] Sect. 4] with all of the same restrictions. Hasker uses a stronger type of communication, and comes up with a more optimistic answer. In that paper if players can send costly messages (letters) to all other players then the folk theorem can be proven. The analysis is only for two player games, and the stage game must be neither one of pure conflict or pure cooperation, but otherwise the results are the same as here.

Another related line of research is the limited information folk theorems. The classic problem in this literature is the imperfect private information problem. Kandori and Matsushima [16] prove a folk theorem in this setting using communication. They use noiseless communication to overcome the noisiness of the original observation. Ben-Porath and Kahneman [4] analyzes a situation where only some players observe what each player does. The motivating example is a large corporation where people only see others in their department. However, that result relies on public declarations and thus would not work in extremely large corporations. This paper shows that if players' payoffs do not depend on people outside their division then there is a folk theorem appropriate for these corporations.

The next section of the paper presents the model. Following this the equilibrium condition is established. Next we illustrate the cost of local information processing by showing that trigger strategies and simple optimal penal codes are not equilibria in the repeated matching game. In this section a simple concrete random matching game is described—the Looped Townsend Turnpike. In section 5 the main theorem is presented. The theorem is also extended to the case where there is some heterogeneity between players, and the problems with mixed strategy equilibria are discussed.

2 A Description of the Model

In a *matching game* there are I populations each with the same number of members, which are at most countable. Call these populations P_i $i \in \{1, 2, 3, ...I\}$, then a matching rule μ is a distribution over deterministic matching rules. A deterministic matching rule is a function $\psi : P_1 \to \times_{k=2}^{I} P_k$ whose projection onto P_k is one to one for all $k \neq 1$. The players matched with player j are denoted $\mu(j)$, and when the group is matched they play a stage game.[1]

Until subsection 4.3 in the stage game all that matters is a player's population, it does not matter which player is involved. This is also true about the strategy and many other times during our analysis. When this is true I will describe a player as i and it should be understood that I am speaking of an arbitrary member of population P_i. When I need to speak of an individual I will refer to her as j. Thus in the stage game player i has a finite action set, A_i, and payoff function, $\pi_i : \times_{k=1}^{I} A_k \to R$. A player's own action will be written first in her payoff function, and the vector of actions taken by the other people in her group will be written as $a_{\mu(i)} \in \times_{k \neq i} A_k$. Without loss of generality (see corollary 2) players use correlated actions from the set $A = \Delta(\times A_i)$. In this paper the minmax is in pure strategies, call this strategy $m^i \in \times A_k$ for population i and normalize payoffs so that $\pi_i(m^i) = 0$. Note that since we are discussing matching games players should have the option of not interacting, and thus the mixed strategy minmax is in pure strategies.

In a *repeated matching game* the interaction above will happen ad infinitum, with players discounting their payoffs between periods by $\delta \in (0, 1)$. Define H^t as the history of the entire game up to period t. This includes the actions of all players (and who they interacted with) in period 0 to $t-1$. Then the matching rule in each period is determined by a *matching regime* μ^r, thus if the set of matching rules is M, $\mu^r : \times_{t=1}^{\infty} H^t \to M$. A *path* is a sequence of action profiles $w = \{a^t\}_{t=1}^{\infty}$ $a^t \in A$. Her payoff from such a path is her discounted value, $v_i : A^\infty \to R$.

The equilibria will be *social norms* $\{Z, \sigma, \tau\}$. The social status of a player of population i in period t is denoted $z_i^t \in Z_i$. $Z = \times_{i=1}^{I} Z_i$ is the *set of social statuses*, and it will be finite. The *social standard of behavior* is a function from social status to action today, and is denoted $\sigma : \times_i Z_i \to A$. The *transition rule* is $\tau = \{\tau_i\}_{i=1}^{I}$. It takes a player's social

[1] An example of an interesting matching rule is to choose one population $i \in \{1, 2, 3, ...I\}$. Then choose players one at a time from this population and ask them to select one person from each of the other populations who is not already chosen. This matching rule is a model of employers hiring employees.

The inverse of this matching rule is also of interest. Select a player from all of the populations except 1 one at a time, and ask her which of the P_1 she wants to be matched with among the $j \in P_1$ that are not already matched with someone of her population. This matching rule is a model of customers choosing a store to buy from.

status—z_i^{t-1}, what action she took last period—a_i^{t-1}, and what action she was supposed to take—σ_i^{t-1}; adds in the same information for her opponents this period and assigns the player a new social status. For simplicity τ will be restricted so that it is not directly a function of a_i^{t-1} and σ_i^{t-1}, instead it is only a function of whether $a_i^{t-1} = \sigma_i^{t-1}$, where it's understood that if σ_i^{t-1} is a correlated action $a_i^{t-1} = \sigma_i^{t-1}$ implies that a_i^{t-1} is the correct action given the outcome of the public randomization device. Note that these social norms use *local information processing* since τ is only affected by $\left\{z_k^{t-1}, a_k^{t-1}, \sigma_k^{t-1}\right\}_{k \in i \cup \mu(i)}$. When players use a social norm they are told $z_{\mu(i)}^t$ after they are matched in period t.

Social norms will have to be sequential equilibria and satisfy two other refinements. Define $FI^t(X)$ as the state where the players in X know H^t, and let $X|y$ be the variable X given that y occurred.

Definition 1 *An equilibrium is* straightforward *if given a matching regime, for all i and for all $X \subseteq \cup_{k=1}^I P_k$ $a_i^t|FI^t(X) = \sigma_i\left(z_i, z_{\mu(i)}\right)$*

Notice that a sequential equilibrium must be a subgame perfect equilibrium if it is straightforward.

Definition 2 *An equilibrium is* universal *if it is an equilibrium for all matching regimes and population sizes.*

To clarify some conventions and terminology, in this paper a *standard repeated game* means a repeated matching game where the same players are always matched together. In an abuse of terminology a *random matching game* is a repeated matching game where the matching regime is independent of the history of the game, and a *repeated matching game* is one where the matching regime can depend on history. When we refer to an *equilibrium* we mean a sequential equilibrium that satisfies straightforwardness and universality. A *static Nash equilibrium* is an equilibrium of the repeated matching game when the discount factor is zero. A *strict path equilibrium* is one where along all paths that can be reached after some H^t the incentive to take the equilibrium action is met with a strict inequality.

2.1 Equilibrium Condition

In this section the most important refinement is universality. The only effect in this section of straightforward is that we don't need to worry about players' information sets or beliefs. Local information processing merely limits the type of social norms we must consider.

However, universality means that a "game" in this paper is equivalent to a large class of games in a standard analysis. In a standard analysis the matching regime and population size are part of the definition of the game. This means that usually the future is a unique path—at worst it has a unique expectation. Here there is no unique future, even in expectation.

One reason universality is required is because an analyst must frequently admit to uncertainty about the matching regime in a large repeated matching game. If he or she faces this uncertainty she will have to do analysis similar to that done here. Given that she must embark on this type of analysis, universality actually makes it as simple as possible. Otherwise he or she must worry about the population size and guess at the specific matching regimes she thinks possible. Under universality none of this matters. All that matters are the payoffs of the stage game and the strategy—or the analysis is no more complex than the strategy.

To clarify exposition we will identify a player by her social status, or $z_i^t \in Z_i$ for $i \in \{1, 2, 3, ...I\}$. Given this convention, we can define the set of possible futures, $W\left(z_i^t, a_i^t, \sigma_i^t\right)$.

Definition 3 *To define* $W\left(z_i^t, a_i^t, \sigma_i^t\right)$ *let*

1. $h^{t,s}$ *be a continuation history for player* z_i^t *between period* t *and* $t+s$ $s \in \{1, 2, 3, ...\infty\}$.
2. $\bar{Z}_{-i}^{t+s}\left(h^{t,s}\right) \subseteq \times Z_l$ $l \in \{1, 2, 3, ...I\} \setminus i$ *be the possible set of statuses in period* $t+s$ *given* $h^{t,s}$

Then let $h_\alpha = \lim_{s\to\infty} h_\alpha^{t,s}$ *for a sequence* $\{h_\alpha^{t,s}\}_{s=1}^{\infty}$ *where* $h_\alpha^{t,1} = \{z_i^t, a_i^t, \sigma_i^t\}$ *and* $h_\alpha^{t,s} = h_\alpha^{t,s-1} \cup z_{-i}$, $z_{-i} \in \bar{Z}_{-i}^{t+s-1}\left(h_\alpha^{t,s-1}\right)$ *then* $W\left(z_i^t, a_i^t, \sigma_i^t\right) = \cup_\alpha h_\alpha$ *for all such* α. *It is defined given* σ_i^t*—the pure action* z_i^t *should take today, and* $a_i^t \in A_i$*—the action she actually takes.*

Notice that this is independent of any given population. In essence since the population can be countable we can construct a large enough population that we can achieve this independence, as the following lemma proves.

Lemma 1 *Assume* $\{Z, \tau, \sigma\}$ *is an equilibrium social norm, then the set of potential continuation paths is* $W\left(z_i^t, a_i^t, \sigma_i^t\right)$.

Proof. Given $h^{t,s}$, there is a population size where for every $z_{-i} \in \bar{Z}_{-i}^{t+s}\left(h^{t,s}\right)$ there can be $I-1$ players who have this vector of statuses. By definition every status in $\bar{Z}_{-i}^{t+s}\left(h^{t,s}\right)$ can be reached by some sequence of play, thus every element can be achieved after a finite number of deviations (in 0 to $t-1$) and matchings. Since $\bar{Z}_{-i}^{t+s}\left(h^{t,s}\right) \subseteq \times_{l\neq i} Z_l$ it is finite, and so for a large enough population it is possible for every $z_{-i} \in \bar{Z}_{-i}^{t+s}\left(h^{t,s}\right)$ to be the statuses of some $I-1$ players.

Second, the set of $h^{t,s}$ is finite for every s. Since $h^{t,1}$ is unique, and $h^{t,s} = h^{t,s-1} \cup z_{-i}$, $z_{-i} \in \bar{Z}_{-i}^{t+s-1}\left(h^{t,s-1}\right)$, $\#\left(H^{t,s}\right) \leq \#\left(H^{t,s-1}\right) * \#\left(\times_{l\neq i} Z_l\right)$.

Now since for each s it only requires a finite population then for all s it only requires a countable population to make every continuation history $h^{t,s}$ is possible. Universal requires that the strategy is an equilibrium with a countable population. Universal also requires that we consider all matching regimes, thus the set of possible continuation histories is the set of the sequences $\{h_\alpha^{t,s}\}$ where $h_\alpha^{t,1} = \{z_i^t, a_i^t, \sigma_i^t\}$ and $h_\alpha^{t,s} = h_\alpha^{t,s-1} \cup z_{-i}$, $z_{-i} \in \bar{Z}_{-i}^{t+s-1}\left(h^{t,s-1}\right)$. Thus every path in $W\left(z_i^t, a_i^t, \sigma_i^t\right)$ must be considered. ■

Remark 1 *If we let the social status be a function of σ_i^t instead of only if $a_i^t = \sigma_i^t$ then the above construction would depend on the population because σ would generate a distribution over all populations, and we would need a countable population to make sure all states in $\bar{Z}_{-i}^{t+s}\left(h^{t,s}\right)$ happen each period.*

While $W\left(z_i^t, a_i^t, \sigma_i^t\right)$ may seem complex, except for very odd strategies this set will be finite and analytically simple. Furthermore, the analyst only cares about one of the limiting elements of this set: either an element of $\underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right) \equiv \arg\inf\left\{v_i\left(w\right)|w \in W\left(z_i^t, a_i^t, \sigma_i^t\right)\right\}$ or an element of $\overline{w}\left(z_i^t, a_i^t, \sigma_i^t\right) \equiv \arg\sup\left\{v_i\left(w\right)|w \in W\left(z_i^t, a_i^t, \sigma_i^t\right)\right\}$. In the next lemma we show that $\underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right)$ and $\overline{w}\left(z_i^t, a_i^t, \sigma_i^t\right)$ are actually subsets of $W\left(z_i^t, a_i^t, \sigma_i^t\right)$.

Lemma 2 *Assume $\{Z, \tau, \sigma\}$ is an equilibrium, then $\underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right) \subseteq W\left(z_i^t, a_i^t, \sigma_i^t\right)$ and $\overline{w}\left(z_i^t, a_i^t, \sigma_i^t\right) \subseteq W\left(z_i^t, a_i^t, \sigma_i^t\right)$.*

Proof. Define $z_i^{t+s}\left(h^{t,s}\right)$ as player z_i^t's status in period $t+s$ given the history $h^{t,s}$. Now consider an arbitrary sequence of paths $\{w^\gamma\}_{\gamma=1}^\infty \subseteq W\left(z_i^t, a_i^t, \sigma_i^t\right)$ such that $\lim_{\gamma\to\infty} v_i\left(w^\gamma\right) = v_i\left(\underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right)\right)$. Let $\Omega = \times_s \times_{h^{t,s}} \bar{Z}_{-i}^{t+s}\left(h^{t,s}\right)$, and endow Ω with the product topology. Since $\bar{Z}_{-i}^{t+s}\left(h^{t,s}\right) \times z_i^{t+s}\left(h^{t,s}\right)$ is finite, it is compact and π_i is continuous on it, thus v_i is continuous Ω and by Tychonoff's theorem, Ω is compact. Thus w.l.o.g. assume $\{w^\gamma\}_{\gamma=1}^\infty$ is a convergent sequence and let $w^* = \lim w^\gamma$,then $w^* \in W\left(z_i^t, a_i^t, \sigma_i^t\right)$. Since v_i is continuous, $v_i\left(w^*\right) = v_i\left(\underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right)\right)$ and $w^* \in \underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right)$. The proof that $\overline{w}\left(z_i^t, a_i^t, \sigma_i^t\right) \subseteq W\left(z_i^t, a_i^t, \sigma_i^t\right)$ is symmetric. ■

For simplicity, define $\underline{v}_i\left(z_i^t, a_i^t, \sigma_i^t\right) = v_i\left(\underline{w}\left(z_i^t, a_i^t, \sigma_i^t\right)\right)$ and $\bar{v}_i\left(z_i^t, a_i^t, \sigma_i^t\right) = v_i\left(\bar{w}\left(z_i^t, a_i^t, \sigma_i^t\right)\right)$. With these lemmas completed the equilibrium condition is immediate.

Proposition 1 *A social norm is an equilibrium if and only if for all $i \in \{1, 2, 3, ...I\}$, for all $\left\{z_i^t, z_{\mu_i}^t\right\} \in \times_{i\in I} Z_i$, for all $\sigma^t \in$support$\left(\sigma\left(z_i^t, z_{\mu_i}^t\right)\right)$ and all $a_i^t \in A_i \backslash \sigma_i^t$*

$$\pi_i\left(a_i^t, \sigma_{\mu(i)}^t\right) + \delta\bar{v}_i\left(z_i^t, a_i^t, \sigma_i^t\right) \leq \pi_i\left(\sigma_i^t, \sigma_{\mu(i)}^t\right) + \delta\underline{v}_i\left(z_i^t, \sigma_i^t, \sigma_i^t\right) \tag{1}$$

Proof. This condition is sufficient since for any possible continuation path w, $v_i(w) \geq \underline{v}_i(z_i^t, \sigma_i^t, \sigma_i^t)$, thus if η is a conditional probability over the set of continuation paths $\int_w v_i(w)\, d\eta \geq \underline{v}_i(z_i^t, \sigma_i^t, \sigma_i^t)$—symmetrically for any η', $\int_w v_i(w)\, d\eta' \leq \bar{v}_i(z_i^t, a_i^t, \sigma_i^t)$. It is necessary since by lemma 1 every element of $W(z_i^t, a_i^t, \sigma_i^t)$ must be considered. ■

In words this states a strategy is an equilibrium if and only if it is when cooperation leads to the worst possible payoff and deviating is rewarded as much as possible. A reader may think that the condition is excessively restrictive. However, in the folk theorem section we will show a endogenous matching regime that actually achieves this, (subsection 4.1). Thus it is possible to give examples where this condition is binding. In fact, this example was developed after the equilibrium condition was established. It was easy to see that it failed the above condition, finding a matching regime that generated these payoffs required more thought.

Note that σ_i^t is compared with $A_i \backslash \sigma_i^t$ but not with itself. Upon inspection of the equilibrium condition the reason is transparent. In order for the inequality to be true when comparing σ_i^t with σ_i^t it must be that $\underline{w}(z_i^t, a_i^t, \sigma_i^t) = \overline{w}(z_i^t, a_i^t, \sigma_i^t)$. With such a strategy who a player meets with literally doesn't matter, for all w and w' in $W(z_i^t, \sigma_i^t, \sigma_i^t)$, $v_i(w) = v_i(w')$. Regardless of who she meets, a player's payoff is always the same. Of course if σ_i^t is a mixed strategy this condition must be satisfied, which is why we do not consider such strategies.

3 The Cost of Local Information Processing.

In this section I will illustrate what assuming local information processing has cost the analysis. This cost will be illustrated with two negative results. First it will be proven that there are no trigger strategies which are equilibria. Second it will be shown that there is not always a simple optimal penal code. Thus the simplest equilibria and the analytically most important equilibria in a standard repeated game are lost.

The primary effect of local information processing is that multiple deviations must now be considered. By local information processing a player's status can only be affected by players she has interacted with in the past. This means that multiple people can deviate and all of them must be handled simultaneously. For trigger strategies this means players can have to punish forever independent of whether they cooperate today or not—thus they will defect. For simple optimal penal codes since each player already gets the worst equilibrium payoff after a deviation simultaneous deviations can give them a lower payoff, and they will deviate again.

Of course both of these strategies would be equilibria if it was not for straightforwardness and universality. However as will be shown only weak versions of both of these refinements are needed. It is the primary assumption that causes the failure.

3.1 The Failure of Trigger Strategies in Repeated Matching Games.

The trigger strategy is a classic strategy commonly used to prove folk theorems. In Okuno-Fujiwara and Postlewaite [18] it is used to prove the first folk theorem for random matching games. In that paper local information processing is assumed but equilibria are not straightforward or universal. In standard repeated games Friedman [9] proved the first folk theorem using trigger strategies. Later papers have used other strategies only to generalize his results. In contrast trigger strategies are not equilibria in repeated matching games.

The easiest way to understand this is to consider a concrete matching game, the *Looped Townsend Turnpike*—a modified *Townsend Turnpike* (Townsend [22]). Imagine there is a circular road, along this turnpike there are n restaurants at n different locations. There are n truckers that go around this turnpike, each going one restaurant forward each period and eating at each restaurant. Label the truckers $c \in \{1, 2, 3, ..., n\}$ and the restaurateurs $r \in \{1, 2, 3, ...n\}$, then in period t the restaurateurs' matching regime is $\mu_r^t = (t + r) \operatorname{mod}_n$[2]. The restaurateur can either produce high quality (H) or low quality (L) food, and the trucker can pay (P) or not pay (N) before the quality is revealed. The normal form game is:

		restaurauteur H	restaurauteur L
trucker	P	$2, 1$	$-2, 2$
	N	$4, -1$	$0, 0$

(2)

This is a prisoners dilemma where $\{P, H\}$ is the socially desirable outcome.[3] For a trigger strategy in this game let the set of social statuses be $Z_r = Z_t = \{0, 1\}$ and everyone's initial status be zero. The transition rule is

$$z_i^t = \tau_i\left(z_i^{t-1}, a_i^{t-1}, \sigma_i^{t-1}\right) = \begin{cases} 1 & \text{if } a_i^{t-1} \neq \sigma_i^{t-1} \text{ in } t-1 \\ z_i^{t-1} & \text{otherwise} \end{cases} \tag{3}$$

The social standard of behavior is

$$\sigma\left(z_r^t, z_c^t\right) = \begin{cases} \{H, P\} \text{ if } z_r^t = z_c^t = 0 \\ \{L, N\} \text{ otherwise} \end{cases} \tag{4}$$

[2] where $(t + r) \operatorname{mod}_n$ is the remainder of $\frac{t+r}{n}$ times n.

[3] Note that cooperation due to the threat of personal enforcement will not work in large populations. Personal enforcement requires $\delta^n \geq \frac{1}{2}$ which will fail for all δ if $n > \frac{\ln \frac{1}{2}}{\ln \delta}$.

In this strategy a player is "good" if her status is zero. As long as her status is good she cooperates (sells high quality food or pays) with all good players she meets, otherwise she trusts no one (L or N).

Now in order to see why this is not an equilibrium, given δ choose k such that

$$\pi_c(N,H) - \pi_c(P,H) < \frac{\delta^{k+1}}{1-\delta}\pi_c(P,H) \tag{5}$$

This k is sufficient so that if a trucker has to cooperate today and punish someone (play $\{N, L\}$) for the next k periods she will cheat and play N today. In this matching game this means the next k restaurateurs have the bad status ($z_r^t = 1$).

By universal, the strategy must be an equilibrium if the restaurateurs population is larger than k. For simplicity assume the next k restaurateurs deviate at once. By straightforward the critical trucker can know of these deviations and will defect—thus this strategy is not an equilibrium.

We could overcome this problem if we modified trigger strategies so they were not *absorbing state social norms.*

Definition 4 *In an* absorbing state social norm *there is a social status* z_i^* $i = \{1, 2, 3, ...I\}$ *after which a player who has status* z_i^* *will never have her status changed in any future.*

In fact no absorbing state social norms that satisfy two reasonable criteria are equilibria.

Definition 5 *A social norm is* minimal *if there is a finite sequence of play such that every* $z_i \in Z_i$ *is the status of some player from population* i *at the end of this sequence of play.*

Definition 6 *A social norm is* time independent *if* Z *is minimal and at any period* t *there is a finite continuation sequence of play such that* $z_i \in Z_i$ *is the status of some player from population* i *at the end of this sequence of play.*

Thus social norms that violate one of these criteria either have "wasted" social statuses floating around, or have a finite initial period in which the strategy is not an absorbing state social norm. Without loss of generality, assume that $\sigma(z^*)$ is a static Nash equilibrium each period.[4]

Lemma 3 *Assume* $\{Z, \tau, \sigma\}$ *is an equilibrium absorbing state social norm which is time independent, then for all* $z' = \left\{z_i'\right\} \in \times_{i \in I} Z_i$ $\sigma\left(z'\right)$ *is a static Nash equilibrium.*

[4] Clearly a player with status z_i^* must play a best response each period, and if there is such a status for every role then $\sigma(z^*)$ must be a nash equilibrium.

Proof. Assume $\sigma(z')$ is not a static Nash equilibrium, and $\sigma_1(z')$ is not a best response. Then since the player of population 1 can be matched with a player with status z_2^* from this period on, $\underline{v}\left(z_1', \sigma_i^{t-1}, \sigma_i^{t-1}\right) \leq \frac{1}{1-\delta}\pi_1(\sigma(z^*))$. Clearly $\bar{v}\left(z_1', a_i^{t-1}, \sigma_i^{t-1}\right) \geq \underline{v}\left(z_1', \sigma_i^{t-1}, \sigma_i^{t-1}\right)$ since if a player deviates today they can also be matched with the person with status z_2^*. Thus the strategy is not an equilibrium. ■

Notice that while it might be very complicated to get the status profiles z' and z_2^* the matching regime given these statuses is very simple. If z' is the initial status profile, all the matching regime has to be is "i went travelling for a while, and now she's coming back home." Furthermore, notice that she might be traveling for a very long time and if at any time she gets some news from home she will deviate before she returns. She only needs one piece of information, is it tenable to assume that she will never get it?

One possible response to this result is to point the finger at *universality*. Certainly—one would argue—trigger strategies would work if the matching regime was "suitably diffuse." This is precisely the case that Kandori [[15] Sect. 4] and Ellison [7] analyze. OFP [18] actually assume a continuum of players, and then extend their argument to games with a large finite population and a diffuse matching regime. Combined with this would have to be an argument that *straightforward* is too strong. Perhaps the weakest possible version would be that players only learn the social status of one player with probability ε each period. But with an absorbing state social norm the information that someone has status z_2^* is permanent, and this small ε probability would build up enough information in finite time.

Another weakening of *straightforward* brings out a different difficulty with equilibria that violate this criteria. Assume that players only know the status of k other players—their "neighbors." In this case a strategy would have to be an equilibrium when a player knows that all k have defected, and thus their patience must be higher than in the standard repeated game. When straightforward is weakened we have to have players patient enough that their information doesn't matter, or they are more patient than informed.

A final response is that we aren't really interested in sequential equilibria, for large societies only equilibria where a small fraction of the population deviates should be of interest. The problem with this is that the matching regime *must* be diffuse. If there is a small group that matches primarily among themselves then the deviations in this neighborhood could spread. While a diffuse matching regime is a reasonable assumption if violations don't matter much, with strategies as susceptible to unraveling as absorbing state social norms the assumption is strong.

In Kandori [15] a refinement not used here was required. This refinement was *global stability*—given everyone follows the equilibrium strategy from any period on eventually the payoff will return to the initial path's payoff. This restriction was intended to rule out "contagion" strategies—a type of absorbing state social norm. However these strategies are not equilibria because of local information processing and straightforwardness. Perhaps no assumptions make an absorbing state social norm a "reasonable" equilibrium of a repeated matching game.

3.2 A random matching game without a simple optimal penal code.

The simple optimal penal code (Abreu [1]) is one of the seminal results in the literature of standard repeated games. The simple optimal penal code is defined by finding the worst equilibrium path for each player. The set of these paths is the simple optimal penal code, and this code makes it easy to find all of the equilibria of the standard repeated game. If a path is an equilibrium, it is an equilibrium when after any deviation the continuation path is the worst equilibrium path for that player. Thus instead of finding a potentially countable set of equilibria with one equilibrium you can find all of the others.

That this construction does not generalize to repeated matching games is not surprising. The result depends on the fact that in any history of the game there can be no more than one deviator that must be punished—simultaneous deviations are ignored. Local information processing makes this impossible in repeated matching games, and thus the result should not generalize. The simplicity of this intuition is best appreciated by analyzing an example. Consider the stage game:

		column			
		H_r	H_c	M_r	M_c
row	H_r	$4,2$	$3,3$	$0,-2$	$1,0$
	H_c	$0,0$	$2,4$	$0,-1$	$0,0$
	M_r	$0,0$	$0,1$	$-4,-1$	$0,0$
	M_c	$-1,0$	$-2,0$	$-5,-5$	$-1,-4$

(6)

and let $\delta = \frac{1}{2}$. This game has a unique worst equilibrium path for each player, which is (for $i \in \{r, c\}$)

$$\underline{w}(i) = \begin{cases} \{M_i, M_i\} & \text{if } i \text{ deviated last period} \\ \{H_i, H_i\} & \text{else} \end{cases} \tag{7}$$

It can be easily verified that if $\underline{w}(i)$ is both the initial path and the path after any deviation by either player then this is an equilibrium. It must be the worst since it's payoff is zero. The proof of uniqueness can be requested from the author.

To construct a social norm that uses the simple optimal penal code in this game, let the set of social statuses be $Z_r = Z_c = \{0, 1, 2\}$, write the transition rule as a two step function

$\tau = \{\tau^n, \tau^x\}$ (for $i \in \{r, c\}$)

$$\tilde{z}_i = \tau_i^n \left(z_i^{t-1}, a_i^{t-1}, \sigma_i^{t-1}\right) = \begin{cases} 2 & \text{if } a_i^{t-1} \neq \sigma_i^{t-1} \ t-1 \\ 1 & \text{if } z_i^{t-1} = 2 \\ z_i^{t-1} & \text{else} \end{cases} \tag{8}$$

$$z_i^t = \tau_i^x \left(\tilde{z}_i, \tilde{z}_{\mu(i)}\right) = \begin{cases} 0 & \text{if } \tilde{z}_i \leq \tilde{z}_{\mu(i)} \\ \tilde{z}_i^t & \text{else} \end{cases}$$

The social standard of behavior for any a is:

$$\sigma \left(z_i^t, z_{\mu i}^t\right) = \begin{cases} \{M_i, M_i\} & \text{if } z_i^t = 2 \\ \{H_i, H_i\} & \text{if } z_i^t = 1 \\ a & \text{else} \end{cases} \tag{9}$$

But this strategy is not an equilibrium independent of a. Note that:

$$\pi_i \left(M_i, M_i\right) + \frac{\delta}{1-\delta} \pi_i \left(H_i, H_i\right) = -4 + \frac{\frac{1}{2}}{1-\frac{1}{2}} 4 = 0 \tag{10}$$

since zero is the individually rational payoff the discounted payoff for player i can not decrease. Since $\pi_i (H_i, H_i)$ is the highest possible payoff in the stage game, if player i receives $\pi_i (M_i, M_i)$ today then she *must* receive $\pi_i (H_i, H_i)$ forever in the future or she will deviate. Now assume that two random players r_1 and c_1 both deviated yesterday and will interact tomorrow. Since r_1 and c_1 will meet tomorrow it is impossible for both of them to expect to play $\{H_i, H_i\}$ tomorrow, and the simple optimal penal code is not an equilibrium. This result can be shown to hold for $\delta \in \left[\frac{1}{2}, \frac{9}{17}\right]$.

Note how weak *straightforward* can be for this result to hold. If each player has an $\varepsilon > 0$ chance of learning the social status of a random player the strategy will not be an equilibrium. *Universal* was hardly used at all, the matching regime only has to allow players who last period be matched next period.

4 The Folk Theorem

The folk theorem is fairly immediate given the equilibrium condition, the only remaining step is to describe the strategy. Before taking this step I will first show the effect of endogenous matching regimes by example. This will be done by showing why the strategies used to prove the folk theorem in Kandori [15] are not equilibria with endogenous matching. This should explain to the reader two important facts. First why endogenous matching regimes matter. Second, why the equilibrium condition is binding. One might think that the matching regimes needed to achieve this must be extremely counter-intuitive, but in the following example the matching regime is based on a preference to reward people who haven't deviated. The only argument could be with the tie-breaking rule, which will be discussed.

4.1 The Effect of Choice Driven Matching.

Consider the following prisoner's dilemma:

$$\begin{array}{ccc} & & \text{column} \\ & & \begin{array}{cc} C_c & D_c \end{array} \\ \text{row} & \begin{array}{c} C_r \\ D_r \end{array} & \begin{array}{|c|c|} \hline 1,2 & 4,-1 \\ \hline -2,2 & 0,0 \\ \hline \end{array} \end{array} \tag{11}$$

The social statuses are $Z_r = Z_c = \{0, 1, 2\}$; the transition rule is:

$$z_i^t = \tau_i \left(z_i^{t-1}, a_i^{t-1}, \sigma_i^{t-1} \right) = \begin{cases} 2 \text{ if } a_i^{t-1} \neq \sigma_i^{t-1} \\ 1 \text{ if } a_i^{t-1} = \sigma_i^{t-1} \text{ and } z_i^{t-1} = 2 \\ 0 \text{ if } a_i^{t-1} = \sigma_i^{t-1} \text{ and } z_i^{t-1} \leq 1 \end{cases} \tag{12}$$

and the social standard of behavior is

$$\sigma \left(z_r^t, z_c^t \right) = \begin{cases} \{C_r, C_c\} \text{ if } z_r^t = 0, z_c^t = 0 \\ \{C_r, D_c\} \text{ if } z_r^t > 0, z_c^t = 0 \\ \{D_r, C_c\} \text{ if } z_r^t = 0, z_c^t > 0 \\ \{D_r, D_d\} \text{ if } z_r^t > 0, z_c^t > 0 \end{cases} \tag{13}$$

where r is a representative row player and c is a representative column player.

Consider a matching game where there are two column players $\{c_1, c_2\}$ and two row players $\{r_1, r_2\}$. The first priority of the matching rule will be to maximize the payoff of the column player with lowest status, and c_1 if $z_{c_1}^t = z_{c_2}^t$. When the column player is indifferent the tie breaking rule will be to interact with the row player who has deviated less, and if this fails then r_1.

Now consider a subgame where $\{c_2, r_1, r_2\}$ all deviated yesterday for the first time. If r_1 cooperates today then her payoff will be:

$$\pi_c (C, D) + \delta \pi_c (C, D) + \delta^2 \pi_c (C, C) + \frac{\delta^3}{1-\delta} \pi_c (C, C) = -1 - \delta + \delta^2 + \frac{\delta^3}{1-\delta} \tag{14}$$

But if she deviates then tomorrow she will be matched with c_2 and her payoff is:

$$\pi_c (D, D) + \delta \pi_c (D, D) + \delta^2 \pi_c (C, D) + \frac{\delta^3}{1-\delta} \pi_c (C, C) = -\delta^2 + \frac{\delta^3}{1-\delta} \tag{15}$$

And she will deviate for all $\delta < 1$. This counter example depends on the tie breaking rule, but is it unreasonable a priori? Unless one can reject the tie breaking rule out of hand, one needs to be concerned about what would happen if someone does use this rule. This was the reason universality was required in the first place—otherwise a social planner must play a guessing game about the matching regime when choosing a strategy.

Notice the minor effect of straightforwardness and how little choice was needed. All that is needed is that the matching game has more than four players, and that one player chooses using the specified preferences. On the other side r_1 only has to know the social status of the people she may be matched with tomorrow, and this information with any positive probability.

4.2 The Primary Result.

The only restriction on the repeated matching games in the proof is that the stage game satisfies the *non-equivalent utilities* condition.

Definition 7 *A stage game satisfies* non-equivalent utilities *(NEU) if there is no i and $k \neq i$ such that $\pi_i(a) = \alpha \pi_k(a) + \beta$, $\alpha \geq 0$.*

This condition is easily verifiable and weak, it fails only in games of pure coordination. Abreu, Dutta and Smith [2] prove a folk theorem for all standard repeated games that satisfy this, making it the weakest sufficient condition in the literature.

However, unlike Abreu et al. [2] this folk theorem will only show that any payoff that dominates the minmax in *pure* strategies can be supported. This is a significant restriction in standard repeated games, less so in matching games. Below I analyze this restriction and conjecture about how and when the folk theorem could be extended.

Abreu, Dutta and Smith show this is equivalent to the existence of asymmetric payoff points. These are action profiles $\left\{b^1, b^2, b^3, ...b^I\right\}$ such that $\pi_i(b^i) < \pi_i\left(b^k\right)$ for all i and k where $k \neq i$. Given any arbitrary action profile a^0 $\pi_i(a^0) > 0$ for all i and this vector of action profiles one can construct a set of vectors $\left\{a^1, a^2, a^3, ...a^I\right\}$ such that $0 < \pi_i(a^i) < \pi_i(a^0)$ and $\pi_i(a^i) < \pi_i\left(a^k\right)$ $k \neq i$.

Given these conventions define the action conditional on social status as:

$$a\left(z_i^t, z_{\mu(i)}^t\right) = \begin{cases} a^0 \text{ if} \forall k \in i \cup \mu(i)\, z_k^t = 0 \\ a^k \text{ if} \exists k \in i \cup \mu(i)\, z_k^t > 0 \end{cases} \tag{16}$$

and the social standard behavior

$$\hat{\sigma}\left(z_i^t, z_{\mu(i)}^t\right) = \begin{cases} m^k & \text{if } \exists k \in i \cup \mu(i)\;\; z_k^t > 1 \\ a\left(z_i^t, z_{\mu(i)}^t\right) & \text{if } \forall k \in i \cup \mu(i)\;\; z_k^t > 1 \end{cases} \tag{17}$$

The transition rule is again defined using a two step function $\hat{\tau} = \{\hat{\tau}^n, \hat{\tau}^x\}$:

$$\tilde{z}_i = \hat{\tau}_i^n \left(z_i^{t-1}, a_i^{t-1}, \sigma_i^{t-1} \right) = \begin{cases} T+1 & \text{if } a_i^{t-1} \neq \sigma_i^{t-1} \\ z_i^{t-1} - 1 & \text{if } a_i^{t-1} = \sigma_i^{t-1} \text{ and } z_i^{t-1} > 1 \\ z_i^{t-1} & \text{if } a_i^{t-1} = \sigma_i^{t-1} \text{ and } z_i^{t-1} \leq 1 \end{cases} \tag{18}$$

$$z_i^t = \hat{\tau}_i^x \left(\tilde{z}_i, \tilde{z}_{\mu(i)} \right) = \begin{cases} 0 & \exists k \in \mu(i) \ \ \tilde{z}_i \leq \tilde{z}_k \\ \tilde{z}_i & \forall k \in \mu(i) \ \ \tilde{z}_i > \tilde{z}_k \end{cases}$$

for $k \in \{1, 2, 3, ..., I\} \backslash i$

In this strategy anyone who deviates is punished for the next T periods and then always play a^i. The critical difference between this and Kandori's strategies is that players are forgiven if they interact with someone who deviated in the same—or later—period.

Define $A^{++} = \{a | a \in A, \forall i, \pi_i(a) > 0\}$.

Theorem 1 *If the stage game satisfies the NEU conditions, then as $\delta \to 1$, every $a^0 \in A^{++}$ can be supported as the equilibrium path of a social norm.*

Proof. We will show there exists a $\underline{\delta}$ such that if $\delta \geq \underline{\delta}$ a^0 is an initial equilibrium path supported by $\hat{\sigma}$, $\hat{\tau}$. We will first find player's worst continuation payoff given that they cooperate. When possible, we will then simplify our analysis by showing that one player's incentives are always worst than another's.

Define $\underline{a}^{-i} \in \arg\min_{k \neq i} \pi_i \left(a^k \right)$, $\underline{m}^{-i} \in \arg\min_{k \neq i} \pi_i \left(m^k \right)$, $\Delta \overline{\pi}_i = \max_{a \in A} \pi_i(a) - \min_{a \in A} \pi_i(a)$ and choose T such that $\Delta \overline{\pi}_i < T \pi_i(a^i)$ for all i. If a player has status zero her worst possible future is to punish someone for the next T periods playing $\underline{m}^{-i}$, and then play $\underline{a}^{-i}$ forever. If a player's status is one there are two possibly worst futures: either the worst for status zero or to play a^i forever. Since $\pi_i(a^i) < \pi_i(\underline{a}^{-i})$

$$\frac{\delta}{1-\delta} \pi_i \left(a^i \right) \leq \delta \frac{1-\delta^T}{1-\delta} \pi_i \left(\underline{m}^{-i} \right) + \frac{\delta^{T+1}}{1-\delta} \pi_i \left(\underline{a}^{-i} \right) \tag{19}$$

for high enough δ, choose $\underline{\delta}$ such that this condition is true for all i. Then playing a^i forever is the worst possible future and a player with status zero will cooperate when a player with status one will. A player with status one will cooperate for high enough δ since $\Delta \overline{\pi}_i < T \pi_i(a^i)$, thus choose $\underline{\delta}$ such that for all i:

$$\Delta \overline{\pi}_i \leq \underline{\delta} \frac{1-\underline{\delta}^T}{1-\underline{\delta}} \pi_i \left(a^i \right) \tag{20}$$

Finally, if a player's status is greater than one and $\delta \geq \underline{\delta}$ then she will cooperate if someone who's status is T will. This person will cooperate since $0 \leq \delta^T \pi_i(a^i)$. ■

Just to be clear, this folk theorem satisfies all of the conditions in Kandori [15] except for global stability. A strategy is *globally stable* if for any finite history H^t, $\lim_{t \to \infty} E\left(v_i^{t+s} | H^t \right) =$

v_i^0—where v_i^{t+s} is the continuation value in period $t+s$ for player i. However this is easily satisfied by the above strategy. All one has to do is have a player play a^i for some $T_1 < \infty$ periods and then revert to playing a^0 forever, thus this folk theorem weakens all of the restrictions of that folk theorem. Critically it holds for choice driven matching regimes, but it also holds for all I player games that satisfy the NEU condition.

At the same time it is almost as weak as Abreu, Dutta and Smith [2]. One difference is that I always require the NEU condition. In two player games Fudenberg and Maskin [10] prove the folk theorem without the NEU condition. Abreu et al. show in general that if you can simultaneously minmax players then the NEU condition is not necessary. In these cases the strategy used punishes all people simultaneously, and this type of strategy will not work in repeated matching games.

Consider a subgame where person 1 deviated yesterday and must be punished, person 2 has never deviated but will be matched with person 1 tomorrow and forever in the future. If the game violates the NEU condition, then $u_2(a) = \alpha u_1(a) + \beta$ $\alpha \geq 0$, to avoid degenerate utility functions assume $\alpha > 0$. This means that 2's continuation payoff can not be more than 1's. How do you punish 2 if they deviate today? If you use the same punishment path as for person 1, then the only difference is the first period's payoff. If you use a harsher punishment, then you are in the same situation next period with the roles of one and two reversed.

I have found that one can prove a folk theorem over the *payoff* space in two player games since the highest payoff must be a Nash equilibrium. The interested reader can request this proof from the author, but the general conjecture is that the folk theorem does not hold if the NEU condition is not satisfied.

4.3 Durability to Payoff Perturbations.

The folk theorem as presented is missing one important characteristic of a matching game. Since players' payoffs are the same regardless of who they are matched with they could do just as well by always interacting with the same people. The motivation of matching has to be that payoffs depend on who one interacts with. In general, this would be a stochastic repeated matching game and this paper is not about stochastic games. While stochastic repeated games have been analyzed (Dutta [6]) what can be done when stochastic payoffs and matching are combined is uncertain and left for future research.

However, I will now present an environment where matching is beneficial and the folk theorem holds. In this game we will only slightly perturb players' payoffs. Enough so that there is a gain to matching, but not so much that the methods above don't work.

Definition 8 *In the* perturbed repeated matching game $\{A_i, \pi_i, \delta\}_{i=1}^I$ *for every player* j

of population i:

$$\pi_j(a) = \pi_i(a) + (1-\delta)\rho_j\left(a, \mu_j, t\right)$$

and that $\left|\rho_j\left(a, \mu_j, t\right)\right| \leq \frac{\bar{\rho}}{2}$ *for all* $i \in \{1, 2, 3, ...I\}$, $t \in \{0, 1, 2, ...\}$ *and* μ_j. *Only player* j *knows* $\rho_j(\cdot)$ *though* $\bar{\rho}$ *is common knowledge.*

Note that these payoffs decrease if the frequency of interaction increases. There are several other normalizations that could be done. One could assume $\pi_j(a) = \pi_i(a) + \rho_j\left(a, \mu_j, t\right)$ if players only knew the realizations of $\rho_j(\cdot)$ from t to $t+s$, $s < \infty$, and the $\rho_j(\cdot, \cdot, t)$ are iid.

Regardless of the normalization the intuition is the same. Assume δ is high enough such that there is a strict path equilibrium supporting a^0 in the unperturbed game. Since the equilibrium is strict, if the payoff from cooperating is decreased by $\frac{\varepsilon}{2}$ and the payoff from cheating is increased by $\frac{\varepsilon}{2}$ it will not affect anyone's incentives. Thus their is a strategy supporting a^0 in the perturbed game for small perturbations.

Corollary 1 *Given* a^0 *assume* $\delta > \underline{\delta}$ *then there is a* ρ^* *such that if* $\bar{\rho} \leq \rho^*$ *there is an equilibrium of the perturbed repeated game that supports* a^0.

Proof. Clearly this perturbation can not increase someone's payoffs by deviating by more than $(1-\delta)\frac{\bar{\rho}}{2}$ each period, also it can not reduce someone's payoff from cooperating by more than $(1-\delta)\frac{\bar{\rho}}{2}$. Thus the total discounted effect on both values can not be more than $\frac{\bar{\rho}}{2}$. Then the definition of ρ^* is

$$\rho^* = \min\left\{\delta\frac{1-\delta^{T-1}}{1-\delta}\pi_i\left(\underline{m}^{-i}\right) + \frac{\delta^T}{1-\delta}\pi_i\left(\underline{a}^{-i}\right) - \frac{\delta}{1-\delta}\pi_i\left(a^i\right), \delta\frac{1-\delta^{T-1}}{1-\delta}\pi_i\left(a^i\right) - \Delta\overline{\pi}_i, \delta^T\pi_i\left(a^i\right)\right\}$$

and all equilibrium conditions will be satisfied with at least weak inequality. ■

4.4 A Discussion of Mixed Strategy Equilibria.

Previously I mentioned that since matching games should include the possibility of not interacting the minmax will be in pure strategies—unlike in standard repeated games. However this is not the only reason that equilibria with a mixed strategy minmax—or other mixed strategies—should not be considered, here I wish to discuss a second problem with such equilibria.

In general, the mixed strategy which minmaxes a given player is not a static Nash equilibrium. In the standard repeated game this is overcome by changing players' future payoffs to make them just indifferent today. To achieve this indifference in standard repeated games players must at least have common beliefs about each other's payoffs for every future period. This is a palatable assumption in a standard repeated game where a group of players will interact forever, but in a repeated matching game two players could interact only once—is the assumption still palatable?

To achieve this indifference in a repeated matching game means that in a large society all players have identical beliefs about the distribution of the $\rho_j\left(a, \mu_j, t\right)$ for all j and all t. This assumption is equivalent to stating that you and all other people in your country have common beliefs about what each other's incomes are—furthermore for any future date everyone has common beliefs today about what the distribution will be then. Assuming player's payoffs are independent of time and who is matched together is but a subtler version of assuming common beliefs over $\rho_j\left(a, \mu_j, t\right)$.

And there is a second dimension to the problem. Since the matching regime might be affected by a player's action it might depend on which action in the support of the mixed strategy is played. Thus a player must be indifferent over *who they interact with.* This problem is why every constant path social norm in the following stage game has a limiting average payoff of one.

	L	R
U	$0, 1$	$1, 1$
M	$-1, 4$	$-1, -4$
D	$-1, -4$	$-1, 4$

(21)

A *constant path social norm* is one where given any history of play and $\left\{z_{\mu(i)}^{t+s}, z_{\mu(i)}^{t+s+1}\right\}$ that are consistent with the strategy being an equilibrium:

$$\#\left(s | s \in \{1, 2, 3, ...\infty\}, \sigma_i\left(z_i^{t+s}, z_{\mu(i)}^{t+s}\right) \neq \sigma_i\left(z_i^{t+s+1}, z_{\mu(i)}^{t+s+1}\right)\right) < \infty \tag{22}$$

this allows any actions in the next T periods but rules out strategies where the action profile cycles. The limiting average payoff is $\lim_{\delta \to 1} (1-\delta) v_i$. The proof can be requested from the author.

Understanding what can be done with social norms that are not constant path will provide insight on how to prove the folk theorem with a mixed strategy minmax. In the proof the reason that all lower payoffs fail is that a player could learn *step by step* that they will have to minmax forever. Say that i is minmaxing today. She looks around and notices that if she plays the wrong action in the mixed strategy then tomorrow she will have to minmax. In order to get her indifferent between this action and her others she will have to be rewarded if she minmaxes tomorrow. Tomorrow the same thing happens, and soon enough the reward she must be promised can not be delivered.

With cyclical equilibria we can stop this chain of reasoning. In essence, players cycle between the payoff one is interested in and a strict Nash equilibrium. After this phase if a column player has to punish a row player by mixing all previous changes in payoff are wiped out. In essence, since strategies that use a mixed punishment in every period are impossible you mix these strategies with static Nash equilibria. As $\delta \to 1$, the frequency of this "silent period" can decrease and any payoff on the interior of the payoff space may be achievable. The author conjectures the folk theorem can be proven at least in games with I Nash equilibria if each population's payoff is minimized on a different equilibrium.

However, as mentioned before the author does not think mixed strategy equilibria exist in any reasonable model that is closer to reality. While a model should be an abstraction if including an important and minimal realistic element changes results drastically then results are suspect. The equilibria in this paper stand up to this criterion, the author does not think mixed strategy equilibria do.

4.5 The Folk Theorem without Correlated Actions.

A final point is that the folk theorem does hold without correlated actions. Since correlating devices are not commonly observed some feel assuming one is a strong assumption. The author would like to point out that Francois Forges [8] did show that players who can play a mixed strategy can create a correlating device (with communication) but would also like to show that the folk theorem holds without this assumption[5]. Fudenberg and Maskin [11] showed that correlated actions can be used without loss of generality in standard repeated games, but one can not automatically assume the proof extends. The difficulty is that without correlated actions the matching regime can give a player a lower payoff than the payoff of any given correlated action, and one must show that these lower payoffs are sufficient for players to be willing to cooperate. Thus the following corollary is included to assure the reader that Fudenberg and Maskin's construction can be extended. For simplicity we will not present detailed strategies, just show that it is possible to construct such strategies.

Corollary 2 (Purification of Correlated Actions) *For any correlated action profile $a^0 \in A$ such that $\pi_i(a) > 0$ for all i there exists a $\tilde{\delta}$ and a path $w^0 = \{a^0(t)\}$ such that $|(1-\delta) v_i^t(w^0) - \pi_i(a^0)| < \varepsilon$ and w^0 is supported as an equilibrium path.*

Proof. Fudenberg and Maskin [11] prove that you can construct a sequence that satisfies $|(1-\delta) v_i^t(w^0) - \pi_i(a^0)| < \varepsilon$ for high enough $\tilde{\delta}$. Also choose $\tilde{\delta}$ high enough that $\pi_i(a^0) - (1-\delta)\varepsilon > 0$.

[5] In this case the matching regime would have to be unaffected by player's messages.

Let $w^i \in \arg\min_{a \in A} \pi_i(a)$, then Abreu, Dutta, and Smith [2] construct the a^i

$$\pi_i\left(a^i\right) = \chi\left[(1-\eta)\,\pi_i\left(w^i\right) + \eta\pi_i\left(b^i\right)\right] + (1-\chi)\,\pi_i\left(a^0\right) \tag{23}$$

for unspecified χ and η such that

$$(1-\eta)\,\pi_i\left(w^i\right) + \eta\pi_i\left(b^i\right) < \min_t (1-\delta)\, v_i^t\left(w^0\right) \tag{24}$$

which is satisfied for small enough η for all players since $\pi_i\left(w^i\right) < \min_t (1-\delta)\, v_i^t\left(w^0\right)$.

Since χ and η are the same for all i, and η only satisfies an inequality constraint, without loss of generality we can assume χ and η are rational. Thus there exists a $\tilde{\delta}$ such that the same sequence $\left\{a^i(t)\right\}$ can be used for all roles, where the difference between $a^i(t)$ and $a^k(t)$ is that when $a^i(t) = w^i$ when $a^k(t) = w^k$ and $a^i(t) = b^i$, $a^k(t) = b^k$. Define $\underline{w}^{-i} \in \arg\min_{k \neq i} \pi_i\left(w^k\right)$ and $\underline{b}^{-i} \in \arg\min_{k \neq i} \pi_i\left(b^k\right)$, and define $\left\{\underline{a}^{-i}(t)\right\}$ as the sequence where $\underline{a}^{-i}(t) = \underline{w}^{-i}$ if $a^i(t) = w^i$ and $\underline{a}^{-i}(t) = \underline{b}^{-i}$ if $a^i(t) = b^i$.

In the strategy used to support $\left\{a^0(t)\right\}$ if a player deviates while being minmaxed it is ignored, all other deviations are dealt with as before. Choose T such that $\Delta\overline{\pi}_i < \min_t \Sigma_{s=1}^T \pi_i\left(a^i(t+s)\right)$ for all i. Defining $\underline{\pi}_i = \min_{a \in A} \pi_i(a)$ choose $\tilde{\delta}$ such that

$$\delta\frac{1-\delta^T}{1-\delta}\underline{\pi}_i \leq \min_t \delta^T \Sigma_{s=1}^{\infty} \delta^s \left(\pi_i\left(\underline{a}^{-i}(t+s)\right) - \pi_i\left(a^i(t+s)\right)\right) \tag{25}$$

$$\Delta\overline{\pi}_i \leq \min_t \Sigma_{s=1}^T \delta^s \pi_i\left(a^i(t+s)\right) \tag{26}$$

■

5 Concluding Thoughts

Thus society can help individuals overcome their short run incentives and act in a cooperative manner. The folk theorem has very simple existence conditions to compensate for the naturally heterogenous nature of the interaction. It only depends on the payoffs of the stage game and the frequency of interaction. This simplicity comes at a cost, trigger strategies do not support cooperative behavior and the simple optimal penal code does not always exist.

However the benefit of the simplicity is significant. A concern of many theorists is the computational complexity of equilibria we describe. Most people (including this theorist) are boundedly rational. OFP [18] point out that following a social norm is much easier than calculating an equilibrium. The essence of a social norm is that you do what you are told. No computation power is required to determine your optimal choice, you follow instructions. The equilibria here take this insight one step further. As long as everyone in the repeated matching game is committed and people's payoffs are not too heterogenous, people never need to know anything except the social status of the person they are interacting with. They can blithely live their lives knowing that whatever happens they should just do as they are told. Thus these equilibria require a minimal amount of computation.

At the same time the generality of the conclusions can seem troubling. After all this analysis shows that if players are patient enough then anything can happen. They don't even have to interact with the same people—one testable restriction in most previous analysis. The author thinks this impression is false. This paper does not prove that societies can cooperate arbitrarily, it shows that they can if they have a well developed mechanism. It should be clear that if the analyst can not point to the mechanism then this paper says nothing. Identifying and understanding the mechanism is an appropriate goal for empirical analysis of social norms. What is indicated is that the mechanism does not have to be run by some government agency, in fact it can be very informal. Examples that illustrate this point abound. Udry [23] found that rural Nigerians did not suffer from credit shortages because they used an informal information network to overcome the moral hazard problem. Greif [12] showed that the Maghibiri traders used a more formalized network—with a centralized information depository—to enable shipping across the Mediterranean during the Middle Ages. Milgrom, North and Weingast [17] found an explicit mechanism in the Champagne fairs. The fairs operated as quality guarantor by banning anyone accused of not trading fairly. Thus they helped trade flourish at the beginning of the Renaissance. In the political science literature Ostrom [19] and others have been extensively studying modern social norms used to overcome public good problems. This literature can also contribute to the theoretic literature by examining how these social norms develop. Ostrom [19] and Greif [13] have done analysis of this type.

The theory of social norms currently has several significant issues that should be addressed. One is allowing for more heterogeneity. Obviously heterogeneity is important for matching games and the amount allowed for here is much less than the amount observed. Dutta [6] has established a folk theorem with weak conditions for stochastic repeated games. While it would perhaps be simple to combine a stochastic game and a matching game, it might be harder to satisfy a condition similar to *universality*. However the motivation for this is as strong as for universality, one can not expect the analyst and the players of the game to know all the details of other players' payoffs. Another problem is that the analysis does not allow for free exit. Free entry can easily be dealt with by the universal restriction, but exit can not be allowed. Empirically this is an important subject, Ostrom [19] finds that limiting free exit is one indicator of a successful social norm. Understanding this point theoretically would be beneficial. A related question is what happens if there are competing social norms?

This paper has developed the understanding of cooperative behavior in market interactions. Many market interactions have their element of moral hazard or agency, and a general theory of how these problems are overcome has long been needed. The traditional solution is to rely on courts, but what if these institutions do not function properly or do not exist? Is the economy dependent on them? The theoretical and empirical literature suggest not; an understanding of this point should be developed.

References

[1] Abreu, Dilip. (1988) "On the Theory of Infinitely Repeated Games with Discounting." *Econometrica*. 56:383-396.

[2] Abreu, Dilip; Prajit K. Dutta; and Lones Smith. (1994) "The Folk Theorem for Repeated Games: A NEU Condition." *Econometrica*. 62:939-48.

[3] Ahn, Illtae and Matti Suominen. (1997) "Word-of-Mouth Communication and Community Enforcement." *mimeo.*

[4] Ben-Porath, Elchanan and Michael Kahneman. (1996) "Communication in Repeated Games with Private Monitoring." *Journal of Economic Theory.* 70:281-297.

[5] Chwe, Michael. (2000). "Communication and Coordination in Social Networks." *Review of Economic Studies.* forthcoming.

[6] Dutta, Prajit K. (1995) "A Folk Theorem for Stochastic Games." *Journal of Economic Theory.* 66:1-32.

[7] Ellison, Glenn. (1994) "Cooperation in the Prisoner's Dilemma with Anonymous Random Matching." *Review of Economic Studies*. 61:567-588

[8] Forges, Francoise. (1990) "Universal Mechanisms." *Econometrica*. 58:1341-1364.

[9] Friedman, James W. (1970) "A Non-cooperative Equilibrium for Supergames." *Review of Economic Studies.* 37:1-12.

[10] Fudenberg, Drew and Eric Maskin. (1986) "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information." *Econometrica* 54:533-554

[11] Fudenberg, Drew, and Eric Maskin. (1991). "On the Dispensability of Public Randomization in Discounted Repeated Games." *Journal of Economic Theory.* 53: 428-438.

[12] Greif, Avner. (1989) "Reputation and Coalitions in Medieval Trade: Evidence on the Maghribi Traders." *Journal of Economic History.* 4:857-882.

[13] Greif, Avner. (1995) "Cultural Beliefs and the Organization of Society: A Historical and Theoretical Reflection on Collectivist and Individualist Societies." *Journal of Political Economy.* 102:912-950.

[14] Hasker, Kevin. (2000) "Cooperation through Communication: A Folk Theorem for

Repeated Matching Games supported by Messages." *mimeo.*

[15] Kandori, Michihiro. (1992) "Social Norms and Community Enforcement." *Review of Economic Studies.* 59: 63-80.

[16] Kandori, Michihiro; and Hitoshi Matsushima. (1998) "Private Observation, Communication and Collusion." *Econometrica.* 66:627-652.

[17] Milgrom, P., D. North and B. Weingast. (1995) "The Role of Institution in the Revival of Trade: The Law Merchant, Private Judges, and the Champagne Fairs." *Economics and Politics.* 2: 1-23

[18] Okuno-Fujiwara, Masahiro; and Andrew Postlewaite. (1995) "Social Norms and Random Matching Games." *Games and Economic Behavior.* 9: 79-109.

[19] Ostrom, Elinor. (1991) "Governing the Commons : the Evolution of Institutions for Collective Action." Cambridge University Press, New York.

[20] Rosenthal, Robert W. (1979). "Sequences of Games with Varying Opponents." *Econometrica.* 47: 1353-1366.

[21] Rosenthal, Robert W., and Henry Landau. (1979) "A Game-Theoretic Analysis of Bargaining with Reputations." Journal of Mathematcial Psychology. 20: 233-255.

[22] Townsend, Robert M. (1980) "Models of Money with Spatially Separated Agents." in *Models of Monetary Economies.* Edited by John H. Kareken and Neil Wallace. Minneapolis: Federal Reserve Bank.

[23] Udry, Christopher. (1994) "Risk and Insurance in a Rural Credit Market: An Empirical Investigation in Northern Nigeria." *Review of Economic Studies.* 61:495-526.