Bootstrap Unit Root Tests in Panels with Cross-Sectional Dependency[1]

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Abstract

We apply bootstrap methodology to unit root tests for dependent panels with N cross-sectional units and T time series observations. More specifically, we let each panel be driven by a general linear process which may be different across cross-sectional units, and approximate it by a finite order autoregressive integrated process of order increasing with T. As we allow the dependency among the innovations generating the individual series, we construct our unit root tests from the estimation of the system of the entire N cross-sectional units. The limit distributions of the tests are derived by passing T to infinity, with N fixed. We then apply bootstrap method to the approximated autoregressions to obtain critical values for the panel unit root tests, and establish the asymptotic validity of such bootstrap panel unit root tests under general conditions. The proposed bootstrap tests are indeed quite general covering a wide class of panel models. They in particular allow for very general dynamic structures which may vary across individual units, and more importantly for the presence of arbitrary cross-sectional dependency. The finite sample performance of the bootstrap tests is examined via simulations, and compared to that of commonly used panel unit root tests. We find that our bootstrap tests perform relatively well, especially when N is small.

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1. Introduction

Recently, nonstationary panels have drawn much attention in both theoretical and empirical research, as a number of panel data sets covering relatively long time periods have become available. Various statistics for testing unit roots and cointegration for panel models have been proposed, and frequently used for testing growth convergence theories, purchasing power parity hypothesis and for estimating long-run relationships among many macroeconomic and international financial series including exchange rates and spot and future interest rates. Such panel data based tests appeared attractive to many empirical researchers, since they offer alternatives to the tests based only on individual time series observations that are known to have low discriminatory power. A number of unit roots and cointegration tests have been developed for panel models by many authors. See Levin and Lin (1992,1993), Quah (1994), Im, Pesaran and Shin (1997) and Maddala and Wu (1996) for some of the panel unit root tests, and Pedroni (1996,1997) and McCoskey and Kao (1998) for the panel cointegration tests available in the current literature. Banerjee (1999) gives a good survey on the recent developments in the econometric analysis of panel data whose time series component is nonstationary.[2]

Since the work by Levin and Lin (1992), a number of unit root tests for panel data have been proposed. Levin and Lin (1992,1993) consider unit root tests for homogeneous panels, which are simply the usual t-statistics constructed from the pooled estimator with some appropriate modifications. Such unit root tests for homogeneous panels can therefore be represented as a simple sum taken over $i = 1, \ldots, N$ and $t = 1, \ldots, T$. They show under cross-sectional independency that the sequential limit of the standard t-statistics for testing the unit root is the standard normal distribution.[3] For heterogeneous panels, the unit root test can no longer be represented as a simple sum since the pooled estimator is inconsistent for such heterogeneous panels as shown in Pesaran and Smith (1995). Consequently the second stage N-asymptotics in the above sequential asymptotics does not work here. Im, Pesaran and Shin (1997) look at the heterogeneous panels and propose unit root tests which are based on the average of the independent individual unit root tests, t-statistics and LM statistics, computed from each individual unit. They show that their tests also converge to the standard normal distribution upon taking sequential limits. Though they allow for the heterogeneity, their limit theory is still based on the cross-sectional independency, which can be quite a restrictive assumption for most of the economic panel data we encounter.

The tests suggested by Levin and Lin (1993) and Im, Pesaran and Shin (1997) are not valid in the presence of cross-correlations among the cross-sectional units. The limit distributions of their tests are no longer valid and unknown when the independency assumption is violated. Indeed, Maddala and Wu (1996) show through simulations that their tests have substantial size distortions when used for cross-sectionally dependent panels. As a way to deal with such inferential difficulty in panels with cross-correlations, they suggest to bootstrap the panel unit root tests, such as those proposed by Levin and Lin (1993), Im, Pesaran and Shin (1997) and Fisher (1933), for cross-sectionally dependent panels. They show through simulations that the bootstrap versions of such tests perform much better, but do not provide the validity of using bootstrap methodology.

In this paper, we apply bootstrap methodology to unit root tests for cross-sectionally dependent panels. More specifically, we let each panel be driven by a general linear process which may differ

[2]Stationary panels have a much longer history and have been intensely investigated by many researchers. The readers are referred to the books by Hsiao (1986), Matyas and Sevestre (1996) and Baltagi (1995) for the literature on the econometric analysis of panel data.

[3]The sequential limit is taken by first passing T to infinity with N fixed and subsequently let N tend to infinity. Regression limit theory for nonstationary panel data is developed rigorously by Phillips and Moon (1999). They show that the limit of the double indexed processes may depend on the way N and T tend to infinity. They formally develop the asymptotics of sequential limit, diagonal path limit (N and T tend to infinity at a controlled rate of the type $T = T(N)$) and joint limits (N and T tend to infinity simultaneously without any restrictions imposed on the divergence rate). Their limit thoery, however, assumes cross-sectional independence.

across cross-sectional units, and approximate it by a finite order autoregressive integrated process of order increasing with T. As we allow the dependency among the innovations generating the individual series, we construct our unit root tests from the estimation of the system consisting of the entire N cross-sectional units. The limit distributions of the tests are derived by passing T to infinity, with N fixed. We then apply the bootstrap method to the approximated autoregressions to obtain the critical values for the panel unit root tests based on the original sample, and establish the asymptotic validity of such bootstrap panel unit root tests under general conditions.

The rest of the paper is organized as follows. Section 2 introduces the unit root tests for cross-sectionally dependent panels based on the original sample, and constructs the bootstrap tests by applying the sieve bootstrap methodology to the sample tests. Also discussed in Section 2 are the practical issues arising from the implementation of the sieve bootstrap methodology and the extension of our method to models with deterministic trends. Section 3 derives the limit theories for the asymptotic tests and establishes asymptotic validity of the sieve bootstrap unit root tests. In Section 4, we conduct simulations to investigate finite sample performance of the bootstrap unit root tests. Section 5 concludes, and mathematical proofs are provided in an Appendix.

2. Unit Root Tests for Dependent Panels

We consider a panel model generated as the following first order autoregressive regression:

$$\triangle y_{it} = \alpha_i y_{i,t-1} + u_{it}, \quad i = 1, \ldots, N;\ t = 1, \ldots, T. \tag{1}$$

As usual, the index i denotes individual cross-sectional units, such as individuals, households, industries or countries, and the index t denotes time periods. We are interested in testing the unit root null hypothesis, $\alpha_i = 0$ for all y_{it} given as in (1), against the alternative, $\alpha_i < 0$ for some y_{it}, $i = 1, \ldots, N$. Thus, the null implies that all y_{it}'s have unit roots, and is rejected if any one of y_{it}'s is stationary with $\alpha_i < 0$. The rejection of the null therefore does not imply that the entire panel is stationary. The initial values $(y_{10}, \ldots, y_{N0})$ of $(y_{1t}, \ldots, y_{Nt})$ do not affect our subsequent asymptotic analysis as long as they are stochastically bounded, and therefore we set them at zero for expositional brevity.

It is assumed that the error term (u_{it}) in the model (1) is given by a general linear process specified as

$$u_{it} = \pi_i(L)\varepsilon_{it} \tag{2}$$

where L is the usual lag operator and $\pi_i(z) = \sum_{k=0}^{\infty} \pi_{i,k} z^k$, for $i = 1, \ldots, N$. Note that we let $\pi_i(z)$ vary across i, thereby allowing heterogeneity in individual serial correlation structures. We also allow for the cross-sectional dependency through the cross-correlation of the innovations $\varepsilon_{it},\ i = 1, \ldots, N$ that generate the errors u_{it}. To define the cross-sectional dependency more explicitly, we define the time series innovation $(\varepsilon_t)_{t=1}^{T}$ by

$$\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})' \tag{3}$$

and denote by $|\cdot|$ the Euclidean norm: for a vector $x = (x_i)$, $|x|^2 = \sum_i x_i^2$, and for a matrix $A = (a_{ij})$, $|A| = \sum_{i,j} a_{ij}^2$. For the development of the asymptotics for the sample statistics and the bootstrapped tests, we assume

Assumption 1 Let (ε_t) be a sequence of iid random variables such that $\mathbf{E}\varepsilon_t = 0$, $\mathbf{E}\varepsilon_t\varepsilon_t' = \Sigma$ and $\mathbf{E}|\varepsilon_t|^r < \infty$ for some $r \geq 4$.

Assumption 2 Let $\pi_i(z) \neq 0$ for all $|z| \leq 1$, and $\sum_{k=0}^{\infty} |k|^s |\pi_{i,k}| < \infty$ for some $s \geq 1$, for all $i = 1, \ldots, N$.

Under Assumptions 1 and 2, we may write the linear process given in (2) as an infinite order autoregressive (AR) process $\alpha_i(L)u_{it} = \varepsilon_{it}$ with $\alpha_i(z) = 1 - \sum_{k=1}^{\infty} \alpha_{i,k} z^k$, and approximate (u_{it}) by a finite order AR process

$$u_{it} = \alpha_{i,1} u_{i,t-1} + \cdots + \alpha_{i,p_i} u_{i,t-p_i} + \varepsilon_{it}^{p_i} \tag{4}$$

where $\varepsilon_{it}^{p_i} = \varepsilon_{it} + \sum_{k=p_i+1}^{\infty} \alpha_{i,k} u_{i,t-k}$. The error in approximating (u_{it}) by a finite order AR process can be made negligible if we increase p_i with T. See Chang and Park (2001) for details. For the order p_i in the AR approximation (4), we assume

Assumption 3 Let $p_i \to \infty$ and $p_i = o((T/\log T)^{1/2})$ as $T \to \infty$, for all $i = 1, \ldots, N$.

Some of the limit theories in the paper can be obtained under weaker conditions. In particular, the iid assumption in Assumption 1 is made to make the usual bootstrap procedure meaningful. All our asymptotics here go through for more general models with martingale difference innovations. See Chang and Park (2001). Assumption 3 is sufficient to establish the consistency of our subsequent bootstrap tests in the weak form, i.e., the convergence of conditional bootstrap distributions in probability. To establish the strong consistency or the a.s. convergence of conditional bootstrap distributions, we need to assume that $p_i/n^{(1/rs)+\delta} \to \infty$ with some $\delta > 0$ for all $i = 1, \ldots, N$.[4] The reader is referred to Chang and Park (1999) for further details.

Using the AR approximation of (u_{it}) given in (4), we write the model (1) as

$$\triangle y_{it} = \alpha_i y_{i,t-1} + \sum_{k=1}^{p_i} \alpha_{i,k} \triangle y_{i,t-k} + \varepsilon_{it}^{p_i} \tag{5}$$

since $\triangle y_{it} = u_{it}$ under the null hypothesis. This can be seen as an autoregression of $\triangle y_{it}$ augmented by $y_{i,t-1}$. Our unit root tests will be based on the above approximated autoregression.[5] For practical implementations, we may choose p_i's using the usual order selection criteria such as Schwartz information criterion (BIC) or Akaike information criterion (AIC).[6] The order selection can be based either on the regression (5) with no restriction on α_i's, or on the approximated AR regression in (4) where α_i's are restricted to be zero. This will not affect our subsequent limit theory.

The augmented autoregression (5) can be written in the following matrix form by taking the individual units, with all their T observations, one after another, viz.

$$\begin{pmatrix} \triangle y_1 \\ \vdots \\ \triangle y_N \end{pmatrix} = \begin{pmatrix} y_{\ell,1} & & 0 \\ & \ddots & \\ 0 & & y_{\ell,N} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} X_1^{p_1} & & 0 \\ & \ddots & \\ 0 & & X_N^{p_N} \end{pmatrix} \begin{pmatrix} \beta_1^{p_1} \\ \vdots \\ \beta_N^{p_N} \end{pmatrix} + \begin{pmatrix} \varepsilon_1^{p_1} \\ \vdots \\ \varepsilon_N^{p_N} \end{pmatrix}$$

or more compactly

$$\triangle y = Y_\ell \alpha + X_p \beta_p + \varepsilon_p \tag{6}$$

where for all $i = 1, \ldots, N$, $y_{\ell,i} = (y'_{i,0}, \ldots, y'_{i,T-1})'$, $\beta_i^{p_i} = (\alpha_{i,1}, \ldots, \alpha_{i,p_i})'$ and $X_i^{p_i} = (x_{i1}^{p_i}, \ldots, x_{iT}^{p_i})'$ with $x_{it}^{p_i\prime} = (\triangle y_{i,t-1}, \ldots, \triangle y_{i,t-p_i})$.

We now present our panel unit root tests based on the original sample, and subsequently construct their bootstrap versions later in this section. Their limit theories will be derived in the next section.

[4] However, the presence of this lower bound for p_i's would prelude using an information criterion to select the order for the approximating AR.

[5] Our regression (5) here may be viewed as an extension of the unit root regression considered in Said and Dickey (1984) to the panel models. However, our assumption on the AR order p_i is substantially weaker than the one used by Said and Dickey (1984), due to the result in Chang and Park (2001).

[6] As for the choice among the selection criteria, BIC might be preferred if (u_{it}) is believed to be generated by a finite autoregression, since it yields a consistent estimator for p_i. If not, AIC may be a better choice, since it leads to an asymptotically efficient choice for the optimal order of some projected infinite order autoregressive process. See Chang and Park (1999) for more discussions on this issue.

2.1 Panel Unit Root Tests

For testing the null hypothesis of the unit roots in $y_t = (y_{1t}, \ldots, y_{Nt})'$ generated by (1) and (2), we first consider the tests based on the system GLS and OLS estimation of the augmented autoregression (6). The F-type test based on the feasible system GLS estimator $\hat{\alpha}_{GT}$ of α in (6) for testing the null $\alpha = 0$ is constructed as

$$F_{GT} = \hat{\alpha}_{GT}' (\mathrm{var}(\hat{\alpha}_{GT}))^{-1} \hat{\alpha}_{GT} = A_{GT}' B_{GT}^{-1} A_{GT} \tag{7}$$

where $\hat{\alpha}_{GT} = B_{GT}^{-1} A_{GT}$,

$$\begin{aligned} A_{GT} &= Y_\ell' (\tilde{\Sigma}^{-1} \otimes I_T) \varepsilon_p - Y_\ell' (\tilde{\Sigma}^{-1} \otimes I_T) X_p \left(X_p' (\tilde{\Sigma}^{-1} \otimes I_T) X_p \right)^{-1} X_p' (\tilde{\Sigma}^{-1} \otimes I_T) \varepsilon_p \\ B_{GT} &= Y_\ell' (\tilde{\Sigma}^{-1} \otimes I_T) Y_\ell - Y_\ell' (\tilde{\Sigma}^{-1} \otimes I_T) X_p \left(X_p' (\tilde{\Sigma}^{-1} \otimes I_T) X_p \right)^{-1} X_p' (\tilde{\Sigma}^{-1} \otimes I_T) Y_\ell \end{aligned}$$

and $\tilde{\Sigma}$ is a consistent estimator of the covariance matrix Σ. The limit distribution for the test F_{GT} is easily derived from the asymptotic behaviors of the components A_{GT} and B_{GT} constituting F_{GT}, and is given in Theorem A.1 in the next section.

On the other hand, the system OLS estimator of α in (6) is given by $\hat{\alpha}_{OT} = B_{OT}^{-1} A_{OT}$, and the OLS-based F-type test for testing $\alpha = 0$ is defined similarly as

$$F_{OT} = \hat{\alpha}_{OT}' (\mathrm{var}(\hat{\alpha}_{OT}))^{-1} \hat{\alpha}_{OT} = A_{OT}' M_{FOT}^{-1} A_{OT} \tag{8}$$

where

$$\begin{aligned} A_{OT} &= Y_\ell' \varepsilon_p - Y_\ell' X_p (X_p' X_p)^{-1} X_p' \varepsilon_p, \quad B_{OT} = Y_\ell' Y_\ell - Y_\ell' X_p (X_p' X_p)^{-1} X_p' Y_\ell, \\ M_{FOT} &= Y_\ell' (\tilde{\Sigma} \otimes I_T) Y_\ell - Y_\ell' X_p (X_p' X_p)^{-1} X_p' (\tilde{\Sigma} \otimes I_T) Y_\ell - Y_\ell' (\tilde{\Sigma} \otimes I_T) X_p (X_p' X_p)^{-1} X_p' Y_\ell \\ &\quad + Y_\ell' X_p (X_p' X_p)^{-1} X_p' (\tilde{\Sigma} \otimes I_T) X_p (X_p' X_p)^{-1} X_p' Y_\ell. \end{aligned}$$

The OLS estimator $\hat{\alpha}_{OT}$ is less efficient than the GLS estimator $\hat{\alpha}_{GT}$ in our context. The OLS-based test F_{OT} in (8) is thus expected to be less powerful than the GLS-based test F_{GT} given in (7). However, we observe in our simulations that F_{OT} often performs better than F_{GT} in finite samples, especially when N is large, i.e., when the dimension of the covariance matrix Σ is large.

To construct a consistent estimator for the covariance matrix Σ, we may estimate the regression

$$u_{it} = \tilde{\alpha}_{i,1}^{p_i} u_{i,t-1} + \cdots + \tilde{\alpha}_{i,p_i}^{p_i} u_{i,t-p_i} + \tilde{\varepsilon}_{it}^{p_i} \tag{9}$$

by single-equation OLS for $i = 1, \ldots, N$, with the unit root restriction $\alpha_i = 0$ imposed. The estimates $\tilde{\alpha}_{i,k}^{p_i}$ are uniformly close to $\alpha_{i,k}$ for $1 \le k \le p_i$, and $(\alpha_{i,k})$ become negligible for $k > p_i$ in the limit as long as we let $p_i \to \infty$.[7] As a result, we may consistently estimate variance and covariance estimates of (ε_{it}) using $(\tilde{\varepsilon}_{it}^{p_i})$. This is shown in Park (1999, Lemma 3.1). Of course, one may obtain consistent fitted residuals by estimating the unrestricted regession (5). This again will not affect our limit theory. From $(\tilde{\varepsilon}_{it}^{p_i})$, form the time series residual vectors

$$\tilde{\varepsilon}_t^p = (\tilde{\varepsilon}_{1t}^{p_1}, \ldots, \tilde{\varepsilon}_{Nt}^{p_N})' \tag{10}$$

for $t = 1, \ldots, T$. We then estimate Σ by $\tilde{\Sigma} = T^{-1} \sum_{t=1}^T \tilde{\varepsilon}_t^p \tilde{\varepsilon}_t^{p\prime}$. Notice that

$$\tilde{\Sigma} = \frac{1}{T} \sum_{t=1}^T \varepsilon_t^p \varepsilon_t^{p\prime} + o_p(1) = \frac{1}{T} \sum_{t=1}^T \varepsilon_t \varepsilon_t' + o_p(1) = \mathbf{E} \varepsilon_t \varepsilon_t' + o_p(1)$$

[7] Under Assumptions 1–3, we have $\max_{1 \le k \le p_i} |\tilde{\alpha}_{i,k}^{p_i} - \alpha_{i,k}| = O((\log T/T)^{1/2}) + o(p_i^{-s})$ a.s., and $\sum_{k=p_i+1}^\infty \alpha_{i,k} = o(p_i^{-s})$.

where the second equality follows from Lemma A1 (c) in Appendix. We use $(\tilde{\Sigma} \otimes I_T)$ as an estimator for the variance of the regression error in (6).

The F-type tests F_{GT} and F_{OT} considered here are two-tailed tests which reject the null $\alpha_i = 0$ for all i when $\alpha_i \neq 0$ for some i. Hence, they reject the null of the unit roots not only against the stationarity $\alpha_i < 0$ but also against the explosive cases with $\alpha_i > 0$ for some i. This will have a negative effect on the powers of the tests.

The framework within which we may effectively deal with the aforementioned problem has been recently developed by Andrews (1999).[8] To deal with the problem, we may replace zeros for the members of $\hat{\alpha}_{GT}$ and $\hat{\alpha}_{OT}$ which have positive values. This can be easily carried out by multiplying element by element the estimators $\hat{\alpha}_{GT} = (\hat{\alpha}_{GT,1}, \ldots, \hat{\alpha}_{GT,N})'$ and $\hat{\alpha}_{OT} = (\hat{\alpha}_{OT,1}, \ldots, \hat{\alpha}_{OT,N})'$ respectively by the N-dimensional indicator functions $1\{\hat{\alpha}_{GT} \leq 0\}$ and $1\{\hat{\alpha}_{OT} \leq 0\}$. Denote by $.*$ the element by element multiplication, and use this to modify the estimators $\hat{\alpha}_{GT}$ and $\hat{\alpha}_{OT}$ as follows

$$\hat{\alpha}_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\} = \begin{pmatrix} \hat{\alpha}_{GT,1} 1\{\hat{\alpha}_{GT,1} \leq 0\} \\ \vdots \\ \hat{\alpha}_{GT,N} 1\{\hat{\alpha}_{GT,N} \leq 0\} \end{pmatrix}, \quad \hat{\alpha}_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\} = \begin{pmatrix} \hat{\alpha}_{OT,1} 1\{\hat{\alpha}_{OT,1} \leq 0\} \\ \vdots \\ \hat{\alpha}_{OT,N} 1\{\hat{\alpha}_{OT,N} \leq 0\} \end{pmatrix}.$$

We now define new statistics, which we call K-statistics. From the modified GLS estimator above, we define the GLS-based K-statistic K_{GT} as follows

$$\begin{aligned} K_{GT} &= (\hat{\alpha}_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\})' (\mathrm{var}(\hat{\alpha}_{GT}))^{-1} (\hat{\alpha}_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\}) \\ &= (A_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\})' B_{GT}^{-1} (A_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\}) \end{aligned} \tag{11}$$

and similarly construct the OLS-based K-statistic K_{OT} from the modified OLS estimator as

$$\begin{aligned} K_{OT} &= (\hat{\alpha}_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\})' (\mathrm{var}(\hat{\alpha}_{OT}))^{-1} (\hat{\alpha}_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\}) \\ &= (A_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\})' M_{FOT}^{-1} (A_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\}). \end{aligned} \tag{12}$$

The K-statistics constructed as above are essentially one-sided tests, since they effectively eliminate the probability of rejecting the null against the explosive alternatives. Therefore they are expected to improve the power properties of the corresponding two-tailed F-type tests for testing of the unit root null against the one-way stationary alternatives.

For the test of the unit root, we are testing $\alpha_i = 0$ for all i. Therefore, we are essentially looking at a homogeneous panel, as far as testing of the null hypothesis is concerned. If the AR coefficients α_i's in our original model (1) are homogeneous, i.e., $\alpha_1 = \cdots = \alpha_N = \alpha$, then the corresponding augmented AR in matrix form is given by

$$\triangle y = y_\ell \alpha + X_p \beta_p + \varepsilon_p \tag{13}$$

which is the same as the augmented AR in matrix form for the original heterogeneous model (6), except that here we have an $(NT \times 1)$-vector $y_\ell = (y_{\ell,1}', \ldots, y_{\ell,N}')'$ in the place of the $(NT \times N)$-matrix Y_ℓ and the parameter α is now a scalar instead of an $(N \times 1)$-vector.

It is natural to consider the t-statistics for testing the null hypothesis of the unit roots in the homogeneous model (13), since the parameter α to be tested is now a scalar. Here we do not allow for the heterogeneity of the AR coefficient, as in Levin and Lin (1992,1993). Obviously, the unit root test based on the homogeneous panel (13) is valid, since the model is correctly specified under the null hypothesis of the unit roots. The homogeneous panel, however, may not provide appropriate modellings under the alternative hypothesis, and this may have an adverse effect on the power of

[8] Here we consider testing $\alpha_i = 0$ against $\alpha_i < 0$, and the parameter set is given by $\alpha_i \leq 0$ for each cross-sectional unit $i = 1, \ldots, N$. The value of α_i under the null hypothesis is therefore on the boundary of the parameter set.

the tests. However, if the panel under consideration is believed to be homogeneous, we may use the one-sided t-type tests, which have a clear advantage over the two-tailed F-type tests constructed from the heterogeneous panels.

The OLS and GLS based t-statistics are constructed from the GLS and OLS estimators of the scalar parameter α in the homogeneous augmented AR (13) and are given by

$$t_{GT} = a_{GT} b_{GT}^{-1/2}, \quad t_{OT} = a_{OT} M_{tOT}^{-1/2} \tag{14}$$

where

$$\begin{aligned} a_{GT} &= y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p - y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p)^{-1}X_p'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p \\ b_{GT} &= y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)y_\ell - y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p)^{-1}X_p'(\tilde{\Sigma}^{-1} \otimes I_T)y_\ell \\ a_{OT} &= y_\ell'\varepsilon_p - y_\ell'X_p(X_p'X_p)^{-1}X_p'\varepsilon_p \\ M_{tOT} &= y_\ell'(\tilde{\Sigma} \otimes I_T)y_\ell - 2y_\ell'X_p(X_p'X_p)^{-1}X_p'(\tilde{\Sigma} \otimes I_T)y_\ell \\ &\quad + y_\ell'X_p(X_p'X_p)^{-1}X_p'(\tilde{\Sigma} \otimes I_T)X_p(X_p'X_p)^{-1}X_p'y_\ell. \end{aligned}$$

Our analysis can be easily extended to the models with heterogeneous fixed effects and individual deterministic trends. Suppose the series (z_{it}) with a nonzero heterogeneous fixed effiect is given by

$$z_{it} = \mu_i + y_{it} \tag{15}$$

or with an individual linear time trend by

$$z_{it} = \mu_i + \delta_i t + y_{it} \tag{16}$$

where the stochastic component (y_{it}) is generated as in our earlier model (1). Then for testing the presence of the unit roots in $(y_{1t}, \ldots, y_{Nt})$ we may construct the panel unit root tests similarly from the regression (5) defined with the fitted values, (y_{it}^μ) or (y_{it}^τ), of y_{it} obtained from the preliminary regressions (15) or (16) for $i = 1, \ldots, N$.

As we show in Section 3.1, the limit distributions of the panel unit root tests developed here depend on various nuisance parameters that represent cross-correlations among the individual cross-sectional units. Hence, the inference based directly on such tests are not possible. In the following section, we now propose bootstrapping the tests developed here to deal with the nuisance parameter problems in their limit distributions and to provide a valid basis for inference based on the panel unit root tests for dependent panels.

2.2 Bootstrap Panel Unit Root Tests

In this section, we consider the sieve bootstraps for the various panel unit root tests, F_{GT}, F_{OT}, K_{GT}, K_{OT}, t_{GT} and t_{OT} considered in Section 2.1. Throughout the paper we use the conventional notation $*$ to signify the bootstrap samples, and use $\mathbf{P}^*$ and $\mathbf{E}^*$ to denote, respectively, the probability and expectation conditional upon the realization of the original sample. While constructing the bootstrapped tests, we also discuss various issues and problems arising in practical implementation of the sieve bootstrap methodology.

To construct the bootstrapped tests, we first generate the bootstrap samples (ε_{it}^*), (u_{it}^*) and (y_{it}^*). For the generation of (ε_{it}^*), we need to make sure that the dependence structure among cross-sectional units, $i = 1, \ldots, N$, is preserved. To do so, we generate the N-dimensional vector $(\varepsilon_t^*) = (\varepsilon_{1t}^*, \ldots, \varepsilon_{Nt}^*)'$ by resampling from the centered residual vectors $(\tilde{\varepsilon}_t^p)$ defined in (10) from the fitted autoregression (9). That is, obtain (ε_t^*) from the empirical distribution of $(\tilde{\varepsilon}_t^p - T^{-1}\sum_{t=1}^T \tilde{\varepsilon}_t^p)$, $t =$

$1, \ldots, T$. The bootstrap samples (ε_t^*) constructed as such will, in particular, satisfy $\mathbf{E}^*\varepsilon_t^* = 0$ and $\mathbf{E}^*\varepsilon_t^*\varepsilon_t^* = \tilde{\Sigma}$.[9]

Next, we generate (u_{it}^*) recursively from (ε_{it}^*) as

$$u_{it}^* = \tilde{\alpha}_{i,1}^{p_i} u_{i,t-1}^* + \cdots + \tilde{\alpha}_{i,p_i}^{p_i} u_{i,t-p_i}^* + \varepsilon_{it}^* \tag{17}$$

where $(\tilde{\alpha}_{i,1}^{p_i}, \ldots, \tilde{\alpha}_{i,p_i}^{p_i})$ are the coefficient estimates from the fitted regression (9). Initialization of (u_{it}^*) is unimportant for our subsequent theoretical development, though it may play an important role in finite samples.[10] The coefficient estimates $(\tilde{\alpha}_{i,1}^{p_i}, \ldots, \tilde{\alpha}_{i,p_i}^{p_i})$ used in (17) may be obtained from estimating (9) by the Yule-Walker method instead of the OLS. The two methods are asymptotically equivalent. However, in small samples the Yule-Walker method may be preferred to the OLS, since it always yields an invertible autoregression, thereby ensuring the stationarity of the process (u_{it}^*). See Brockwell and Davis (1991, Sections 8.1 and 8.2). However, the probability of having the noninvertibility problem in the OLS estimation becomes negligible as the sample size increases.

Finally, obtain (y_{it}^*) by taking partial sums of (u_{it}^*), viz., $y_{it}^* = y_{i0}^* + \sum_{k=1}^t u_{ik}^*$ with some initial value y_{i0}^*. Notice that the bootstrap samples (y_{it}^*) are generated with the unit root imposed. The samples generated according to the unrestricted regression (1) will not necessarily have the unit root property, and this will make the subsequent bootstrap procedure inconsistent as shown in Basawa *et al* (1991). The choice of the initial value y_{i0}^* does not affect the asymptotics as long as it is stochastically bounded. Therefore, we simply set it equal to zero for the subsequent analysis in this section.

To construct the bootstrapped tests, we consider the following bootstrap version of the augmented autoregression (5) which was used to construct the sample test statistics

$$\triangle y_{it}^* = \alpha_i y_{i,t-1}^* + \sum_{k=1}^{p_i} \alpha_{i,k} \triangle y_{i,t-k}^* + \varepsilon_{it}^* \tag{18}$$

and write this in matrix form as

$$\triangle y^* = Y_\ell^* \alpha + X_p^* \beta_p + \varepsilon^* \tag{19}$$

where the variables, y^*, Y_ℓ^*, X_p^* and ε^*, are defined with the bootstrapped samples in the exactly same manner as their original sample counterparts y, Y_ℓ, X_p and ε given below (6).

We test for the unit root hypothesis $\alpha = 0$ in (19), using the bootstrap versions of the F-type tests that are defined analogously as the sample F-type tests considered earlier in (7) and (8). The bootstrap GLS and OLS based F-tests are constructed from the GLS and OLS estimators of α in the bootstrap augmented AR regression (19), and are given explicitly as

$$F_{GT}^* = A_{GT}^{*\prime} B_{GT}^{*-1} A_{GT}^*, \quad F_{OT}^* = A_{OT}^{*\prime} M_{FOT}^{*-1} A_{OT}^* \tag{20}$$

where the components, A_{GT}^*, B_{GT}^*, A_{OT}^*, and M_{FOT}^*, are defined analogously as their sample counterparts, A_{GT}, B_{GT}, A_{OT} and M_{FOT}, given below (7) and (8). They are exactly the same except that the bootstrap samples, Y_ℓ^*, X_p^* and ε^*, are used in the places of their original sample counterparts, Y_ℓ, X_p and ε.

[9] Of course, we may resample ε_{it}^*'s individually from the $\tilde{\varepsilon}_{it}^{p_i}$'s for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. In this case, preserving the original correlation structure among the cross-sectional units needs more care. We basically need to pre-whiten $\tilde{\varepsilon}_{it}^{p_i}$'s before resampling, and then re-color the resamples to recover the correlation structure. More specifically, we first pre-whiten $\tilde{\varepsilon}_{it}^{p_i}$'s by pre-multiplying $\tilde{\Sigma}^{-1/2}$ to $\tilde{\varepsilon}_t^p = (\tilde{\varepsilon}_{1t}^{p_1}, \ldots, \tilde{\varepsilon}_{Nt}^{p_N})'$, for $t = 1, \ldots, T$. Next, generate ε_{it}^*'s by resampling from the pre-whitened $\tilde{\varepsilon}_{it}^{p_i}$'s, and then re-color them by pre-multiplying $\tilde{\Sigma}^{1/2}$ to $\varepsilon_t^* = (\varepsilon_{1t}^*, \ldots, \varepsilon_{Nt}^*)'$ to restore the original dependence structure.

[10] We may use the first p_i-values of (u_{it}) as the initial values of (u_{it}^*). The bootstrap samples (u_{it}^*) generated as such, however, may not be stationary processes. Alternatively, we may generate a larger number, say $T + M$, of (u_{it}^*) and discard first M-values of (u_{it}^*). This will ensure that (u_{it}^*) become more stationary. In this case the initialization becomes unimportant, and we may therefore simply choose zeros for the initial values.

We note that the bootstrap F-statistics F^*_{GT} and F^*_{OT} given in (20) also involve the covariance matrix estimator $\tilde{\Sigma}$, which is defined below (10). The estimate $\tilde{\Sigma}$ is the population parameter for the bootstrap samples (ε^*_t), which corresponds to Σ for the original samples (ε_t). We may of course use the bootstrap estimate $\tilde{\Sigma}^*$, say, for the construction of the statistics F^*_{GT} and F^*_{OT} for each bootstrap iteration. The two versions of the bootstrap tests are asymptotically equivalent at least for the first order asymptotics, and we use $\tilde{\Sigma}$ in the construction of the bootstrap tests for convenience.[11]

The bootstrap K-statistics are constructed from the bootstrap samples in the analogous manner in which the sample K-statistics are defined in (11) and (12). They are defined as

$$K^*_{GT} = (A^*_{GT} .* 1\{\hat{\alpha}^*_{GT} \leq 0\})' B^{*-1}_{GT} (A^*_{GT} .* 1\{\hat{\alpha}^*_{GT} \leq 0\}) \tag{21}$$

$$K^*_{OT} = (A^*_{OT} .* 1\{\hat{\alpha}^*_{OT} \leq 0\})' M^{*-1}_{FOT} (A^*_{OT} .* 1\{\hat{\alpha}^*_{OT} \leq 0\}) \tag{22}$$

where $\hat{\alpha}^*_{GT} = B^{*-1}_{GT} A^*_{GT}$ and $\hat{\alpha}^*_{OT} = B^{*-1}_{OT} A^*_{OT}$ are the bootstrap counterparts to the GLS and OLS estimators $\hat{\alpha}_{GT}$ and $\hat{\alpha}_{OT}$ estimated from the sample regression (6).

The bootstrap t-statistics are also constructed in an analogous manner as we constructed the sample t-statistics, t_{GT} and t_{OT}, in Section 2.1. Thus, we consider the homogeneous panel of the bootstrap samples, with $\alpha_1 = \cdots = \alpha_N = \alpha$ imposed, and compute the t-statistics from the corresponding bootstrap augemented AR, which is written in matrix form as

$$\triangle y^* = y^*_\ell \alpha + X^*_p \beta_p + \varepsilon^* \tag{23}$$

The variables appearing in the above regression are defined in the same way as in the augmented AR in matrix form for the bootstrap heterogeneous model (19), except that here we have an $(NT \times 1)$-vector $y^*_\ell = (y^{*\prime}_{\ell,1}, \ldots, y^{*\prime}_{\ell,N})'$ in the place of the $(NT \times N)$-matrix Y^*_ℓ and the parameter α is now a scalar instead of an $(N \times 1)$-vector.

The bootstrapped GLS and OLS based t-statistics are based on the GLS and OLS estimator of α in the homogeneous augmented AR (23), and are given by

$$t^*_{GT} = a^*_{GT} b^{*-1/2}_{GT}, \quad t^*_{OT} = a^*_{OT} M^{*-1/2}_{tOT} \tag{24}$$

where a^*_{GT}, b^*_{GT}, a^*_{OT}, and M^*_{tOT} are constructed from the bootstrap samples y^*_ℓ, X^*_p and ε^*, and defined in the same manner as their sample counterparts, a_{GT}, b_{GT}, a_{OT} and M_{tOT}, given below (14).

We now outline how our bootstrap tests can be implemented in practice. For illustration, we consider the boostrap test F^*_{GT}. To implement the test, we repeat the bootstrap sampling for the given original sample and obtain $c^*_T(\lambda)$ such that $\mathbf{P}^* \{F^*_{GT} \leq c^*_T(\lambda)\} = \lambda$ for any prescribed size level λ. The bootstrap test F^*_{GT} rejects the unit root null hypothesis if $F_{GT} \leq c^*_T(\lambda)$. In Section 3.2 below, it will be shown under appropriate conditions that the bootstrap panel unit root tests considered here are asymptotically valid, i.e., they have asymptotic size λ.

3. Statistical Theories

3.1 Limit Theories for Panel Unit Root Tests

It is well known that an invariance principle holds for a partial sum process of (ε_t) defined in (3) under Assumption 1. That is,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[T\cdot]} \varepsilon_t \to_d B \tag{25}$$

[11] The bootstrap tests based on the bootstrap estimate $\tilde{\Sigma}^*$ may be better for higher order asymptotics, since they more closely mimic the sample statistics than the bootstrap tests based on the population parameter $\tilde{\Sigma}$. The statistics considered in the paper are, however, non-pivotal and therefore the higher order asymptotics are irrelevant here.

as $T\to\infty$, where $B=(B_1,\ldots,B_N)'$ is an N-dimensional Brownian motion with covariance matrix Σ, and $[x]$ denotes the maximum integer which does not exceed x.

For the development of our asymptotics, we may conveniently use the Beveridge-Nelson representation for (u_{it}) given in (2) as

$$u_{it}=\pi_i(1)\varepsilon_{it}+(\bar u_{i,t-1}-\bar u_{it}) \tag{26}$$

where $\bar u_{it}=\sum_{k=0}^\infty \bar\pi_{i,k}\varepsilon_{i,t-k}$ with $\bar\pi_{i,k}=\sum_{j=k+1}^\infty \pi_{i,j}$. Under our condition in Assumption 2, we have $\sum_{k=0}^\infty|\bar\pi_{i,k}|<\infty$ [see Phillips and Solo (1992)] and therefore $(\bar u_{it})$ is well defined both in a.s. and L^r sense [see Brockwell and Davis (1991, Proposition 3.1.1)].

Under the unit root hypothesis $\alpha_1=\cdots=\alpha_N=0$, we may now write

$$y_{it}=\pi_i(1)w_{it}+(\bar u_{i0}-\bar u_{it}) \tag{27}$$

where $w_{it}=\sum_{k=1}^t\varepsilon_{ik}$. Consequently, (y_{it}) behaves asymptotically as the constant $\pi_i(1)$ multiple of (w_{it}). Note that $(\bar u_{it})$ is stochastically of smaller order of magnitude than (w_{it}), and therefore will not contribute to our limit theory.

Let σ_{ij} and σ^{ij} denote, respectively, the (i,j)-elements of the covariance matrix Σ and its inverse Σ^{-1}. The limit theories for the F-type tests, F_{GT} and F_{OT} defined in (7) and (8), are given in

Theorem A.1 Under Assumptions 1 – 3, we have

$$\text{(a) } F_{GT}\to_d Q_{A_G}'Q_{B_G}^{-1}Q_{A_G} \qquad \text{(b) } F_{OT}\to_d Q_{A_O}'Q_{M_{FO}}^{-1}Q_{A_O}$$

as $T\to\infty$, where

$$Q_{A_G}=\begin{pmatrix}\pi_1(1)\sum_{j=1}^N\sigma^{1j}\int_0^1 B_1dB_j\\ \vdots\\ \pi_N(1)\sum_{j=1}^N\sigma^{Nj}\int_0^1 B_NdB_j\end{pmatrix},\quad Q_{A_O}=\begin{pmatrix}\pi_1(1)\int_0^1 B_1dB_1\\ \vdots\\ \pi_N(1)\int_0^1 B_NdB_N\end{pmatrix}$$

$$Q_{B_G}=\begin{pmatrix}\sigma^{11}\pi_1(1)^2\int_0^1 B_1^2 & \ldots & \sigma^{1N}\pi_1(1)\pi_N(1)\int_0^1 B_1B_N\\ \vdots & \vdots & \vdots\\ \sigma^{N1}\pi_N(1)\pi_1(1)\int_0^1 B_NB_1 & \ldots & \sigma^{NN}\pi_N(1)^2\int_0^1 B_N^2\end{pmatrix}$$

and

$$Q_{M_{FO}}=\begin{pmatrix}\sigma_{11}\pi_1(1)^2\int_0^1 B_1^2 & \ldots & \sigma_{1N}\pi_1(1)\pi_N(1)\int_0^1 B_1B_N\\ \vdots & \vdots & \vdots\\ \sigma_{N1}\pi_N(1)\pi_1(1)\int_0^1 B_NB_1 & \ldots & \sigma_{NN}\pi_N(1)^2\int_0^1 B_N^2\end{pmatrix}$$

We note that the limit theories provided in Theorem A.1 and all of our subsequent asymptotic results are derived for non-random p_i's which increase with the sample size. In practice, however, we have to estimate p_i from the data using an order selection criteria such as AIC or BIC. The AIC (BIC) rule selects p_i which minimizes the quantity $\log \hat{\sigma}_T^2 + 2p_i/T$ ($\log \hat{\sigma}_T^2 + p_i \log T/T$). Following the lines in Park (1999), we may indeed show that our limit theories, including the bootstrap consistency results in the next section, continue to hold even for the estimated lag order p_i via AIC or BIC, if we modify our Assumption 3 as $p_i = o((T/\log(T))^{1/2})$.

The limit distributions of the F_{GT} and F_{OT} are nonstandard and depend heavily on the nuisance parameters that define the cross-sectional dependency and the heterogeneous serial dependence. Therefore, it is impossible to perform inference based directly on the tests F_{GT} and F_{OT}.

The limit distributions of the K-statistics can be easily obtained in a manner similar to that used to derive the limit theories for the F-type tests, and are given in

Corollary A.1 Under Assumptions 1 – 3, we have

$$\text{(a) } K_{GT} \to_d (Q_{A_G} .\!* 1\{Q_{B_G}^{-1} Q_{A_G} \leq 0\})' Q_{B_G}^{-1} (Q_{A_G} .\!* 1\{Q_{B_G}^{-1} Q_{A_G} \leq 0\})$$
$$\text{(b) } K_{OT} \to_d (Q_{A_O} .\!* 1\{Q_{B_O}^{-1} Q_{A_O} \leq 0\})' Q_{M_{FO}}^{-1} (Q_{A_O} .\!* 1\{Q_{B_O}^{-1} Q_{A_O} \leq 0\})$$

as $T \to \infty$, where

$$Q_{B_O} = \begin{pmatrix} \pi_1(1)^2 \int_0^1 B_1^2 & \dots & \pi_1(1)\pi_N(1) \int_0^1 B_1 B_N \\ \vdots & \vdots & \vdots \\ \pi_N(1)\pi_1(1) \int_0^1 B_N B_1 & \dots & \pi_N(1)^2 \int_0^1 B_N^2 \end{pmatrix}$$

and the terms Q_{A_G}, Q_{B_G}, Q_{A_O} and $Q_{M_{FO}}$ are defined in Theroem A.1.

As can be seen clearly from the above Corollary, the limit distributions of the K-tests are also nonstandard and depend heavily on the nuisance parameters.

In the following theorem we present the limit theories for the t_{GT} and t_{OT} tests.

Theorem A.2 Under Assumptions 1 – 3, we have

$$\text{(a) } t_{GT} \to_d Q_{a_G} Q_{b_G}^{-1/2} \qquad \text{(b) } t_{OT} \to_d Q_{a_O} Q_{M_{tO}}^{-1/2}$$

as $T \to \infty$, where

$$Q_{a_G} = \sum_{i=1}^N \sum_{j=1}^N \sigma^{ij} \int_0^1 B_i dB_j, \quad Q_{b_G} = \sum_{i=1}^N \sum_{j=1}^N \sigma^{ij} \int_0^1 B_i B_j$$

and

$$Q_{a_O} = \sum_{i=1}^N \pi_i \int_0^1 B_i dB_i, \quad Q_{M_{tO}} = \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} \pi_i \pi_j \int_0^1 B_i B_j$$

The limit processes Q_{a_G}, Q_{b_G}, Q_{a_O} $Q_{M_{tO}}$ appearing in the limit distributions of t_{GT} and t_{OT} are the sums of the individual elements in the corresponding limit processes Q_{A_G}, Q_{B_G}, Q_{A_O} and $Q_{M_{FO}}$ defined in Theorem A.1 for the limit distributions of the tests F_{GT} and F_{OT} that are developed for the heterogenous panels.[12] The limit distributions of the t-statistics t_{GT} and t_{OT} are also nonstandard

[12]Levin and Lin (1992,1993) considers t-statistics for homogeneous panels under cross-sectional independency. Consequently, they can apply N-asymptotics after the limit as T tends to infinity is taken, and derive the limit distribution that is the standard normal. Their theory, however, does not extend to our statistics, since we allow for dependency across cross-sectional units.

and suffer from nuisance parameter dependency, as in the cases with the F-tests and K-statistics. Hence it is not possible to use these statistics for inference as they stand.

The limit theories for the tests given in Theorem A.1, Corollary A.1 and Theorem A.2 extend easily to the models with heterogeneous fixed effects and individual time trends such as those given in (15) and (16), and are given similarly with the following demeaned and detrended Brownian motions

$$B_i^\mu(s) = B_i(s) - \int_0^1 B_i(t)dt$$

and

$$B_i^\tau(s) = B_i(s) + (6s-4)\int_0^1 B_i(t)dt - (12s-6)\int_0^1 tB_i(t)dt$$

in the places of the Brownian motions $B_i(s)$ for $i = 1, \ldots, N$.

3.2 Limit Theories for Bootstrap Panel Unit Root Tests

Here, we establish the consistency of the bootstrapped tests introduced earlier in Section 2.2 and show the asymptotic validity of the tests based on bootstrapped critical values. We will use the symbol $o_p^*(1)$ to signify the bootstrap convergence in probability. For a sequence of bootstrapped random variables Z_T^*, for instance, $Z_T^* = o_p^*(1)$ in $\mathbf{P}$ imply that

$$\mathbf{P}^*\{|Z_T^*| > \delta\} \to 0 \ \text{ in } \mathbf{P}$$

for any $\delta > 0$, as $T \to \infty$. Similarly, we will use the symbol $O_p^*(1)$ to denote the bootstrap version of the boundedness in probability. Needless to say, the definitions of $o_p^*(1)$ and $O_p^*(1)$ naturally extend to $o_p^*(c_T)$ and $O_p^*(c_T)$ for some nonconstant numerical sequence (c_T).

To develop our bootstrap asymptotics, it is convenient to obtain the Beveridge-Nelson representations for the bootstrapped series (u_{it}^*) and (y_{it}^*) similar to those for (u_{it}) and (y_{it}) given in (26) and (27) in the previous section. Let $\tilde{\alpha}_i(1) = 1 - \sum_{k=1}^{p_i} \tilde{\alpha}_{i,k}^{p_i}$. Then it is indeed easy to get

$$u_{it}^* = \frac{1}{\tilde{\alpha}_i(1)}\varepsilon_{it}^* + \sum_{k=1}^{p_i} \frac{\sum_{j=k}^{p_i} \tilde{\alpha}_{i,j}^{p_i}}{\tilde{\alpha}_i(1)}(u_{i,t-k}^* - u_{i,t-k+1}^*) \ = \ \tilde{\pi}_i(1)\varepsilon_{it}^* + (\bar{u}_{i,t-1}^* - \bar{u}_{it}^*)$$

where $\tilde{\pi}_i(1) = 1/\tilde{\alpha}_i(1)$ and $\bar{u}_{it}^* = \tilde{\pi}_i(1)\sum_{k=1}^{p_i}(\sum_{j=k}^{p_i} \tilde{\alpha}_{i,j}^{p_i})u_{i,t-k+1}^*$, and therefore,

$$y_{it}^* = \sum_{k=1}^{t} u_{ik}^* = \tilde{\pi}_i(1)w_{it}^* + (\bar{u}_{i0}^* - \bar{u}_{it}^*)$$

where $w_{it}^* = \sum_{k=1}^{t} \varepsilon_{ik}^*$. From this we may easily derive the limit theories given in the following lemma and Lemma B2 in Appendix that are required for the derivation of the limit distributions for our sieve bootstrap panel unit root tests.

Lemma B1 Under Assumptions 1 – 3, we have

(a) $$\frac{1}{T}\sum_{t=1}^{T} y_{i,t-1}^*\varepsilon_{jt}^* = \tilde{\pi}_i(1)\,\frac{1}{T}\sum_{t=1}^{T} w_{i,t-1}^*\varepsilon_{jt}^* + o_p^*(1)$$

(b) $$\frac{1}{T^2}\sum_{t=1}^{T} y_{i,t-1}^* y_{j,t-1}^* = \tilde{\pi}_i(1)\tilde{\pi}_j(1)\,\frac{1}{T^2}\sum_{t=1}^{T} w_{i,t-1}^* w_{j,t-1}^* + o_p^*(1)$$

We introduce the notation $\to_{d^*}$ for bootstrap asymptotics. For a sequence of bootstrapped statistic (Z_T^*), we write

$$Z_T^* \to_{d^*} Z \text{ a.s.}$$

if the conditional distribution of (Z_T^*) weakly converges to that of Z a.s. as $T \to \infty$. Here it is assumed that the limiting random variable Z has a distribution which does not depend on the original sample realization. We now present the limit theories for the bootstrap tests. The limit theories for the bootstrapped F-tests (F_{GT}^*,F_{OT}^*), K-tests (K_{GT}^*,K_{OT}^*) and t-tests (t_{GT}^*,t_{OT}^*) are given respectively in Theorem B.1, Corollary B.1 and Theorem B.2 below.

Theorem B.1 Under Assumptions 1 – 3, we have as $T \to \infty$,

$$\text{(a) } F_{GT}^* \to_{d^*} Q_{A_G}' Q_{B_G}^{-1} Q_{A_G} \qquad \text{(b) } F_{OT}^* \to_{d^*} Q_{A_O}' Q_{M_{FO}}^{-1} Q_{A_O}$$

in $\mathbf{P}$, where Q_{A_G}, Q_{B_G}, Q_{A_O} and $Q_{M_{FO}}$ are defined in Theorem A.1.

Corollary B.1 Under Assumptions 1 – 3, we have as $T \to \infty$,

$$\text{(a) } K_{GT}^* \to_{d^*} (Q_{A_G} .\!* 1\{Q_{B_G}^{-1} Q_{A_G} \leq 0\})' Q_{B_G}^{-1} (Q_{A_G} .\!* 1\{Q_{B_G}^{-1} Q_{A_G} \leq 0\})$$
$$\text{(b) } K_{OT}^* \to_{d^*} (Q_{A_O} .\!* 1\{Q_{B_O}^{-1} Q_{A_O} \leq 0\})' Q_{M_{FO}}^{-1} (Q_{A_O} .\!* 1\{Q_{B_O}^{-1} Q_{A_O} \leq 0\})$$

in $\mathbf{P}$, where Q_{A_G}, Q_{B_G}, Q_{A_O}, $Q_{M_{FO}}$ and Q_{B_O} are defined in Theorem A.1 and Corollary A.1.

Theorem B.2 Under Assumptions 1 – 3, we have as $T \to \infty$,

$$\text{(a) } t_{GT}^* \to_{d^*} Q_{a_G} Q_{b_G}^{-1/2} \qquad \text{(b) } t_{OT}^* \to_{d^*} Q_{a_O} Q_{M_{tO}}^{-1/2}$$

in $\mathbf{P}$, where Q_{a_G}, Q_{b_G}, Q_{a_O} and $Q_{M_{tO}}$ are defined in Theorem A.2.

The results in Theorem B.1, Corollary B.1 and Theorem B.2 show that the bootstrap tests, (F_{GT}^*, F_{OT}^*), (K_{GT}^*,K_{OT}^*), and (t_{GT}^*,t_{OT}^*) have the same limit distributions as their sample counterparts, (F_{GT},F_{OT}), (K_{GT},K_{OT}) and (t_{GT}^*,t_{OT}^*), which are given respectively in Theorem A.1, Corollary A.1 and Theorem A.2 in Section 3.1. This establishes the asymptotic validity of the boostrap tests F_{GT}^*, F_{OT}^*, K_{GT}^*, K_{OT}^*, t_{GT}^* and t_{OT}^*. We refer to Chang and Park (1999) for a detailed discussion on the issue of asymptotic validity of bootstrap tests in general.

Our bootstrap theories developed here easily extend to the panel unit root tests in models with heterogeneous fixed effects and individual time trends, such as those introduced in (15) and (16). It is straightforward to establish the bootstrap consistency for the tests constructed from the demeaned and detrended series, using the results obtained in this section. The bootstrap tests are therefore valid and applicable also for the models with heterogeneous fixed effects and individual deterministic trends.

4. Simulations

We conduct a set of simulations to investigate the finite sample performance of the bootstrap panel unit root tests, F_{GT}^*, F_{OT}^*, K_{GT}^*, K_{OT}^*, t_{GT}^* and t_{OT}^*, proposed in the paper. For the simulations, we consider two classes of models: (M) the models with heterogeneous fixed effects only and (T) the models with individual time trends as well as fixed effects. More specifically, we consider the models given in (15) and (16) with the series (y_{it}) defined by (1). For each class of models, errors (u_{it}) in (1) are generated as either AR(1) or MA(1) processes, viz.,

$$\text{(AR) } u_{it} = \rho_i u_{i,t-1} + \varepsilon_{it} \qquad \text{(MA) } u_{it} = \varepsilon_{it} + \theta_i \varepsilon_{i,t-1}$$

The innovations $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'$ that generate $u_t = (u_{1t}, \ldots, u_{Nt})'$ are drawn from an N-dimensional multivariate normal distribution with mean zero and covariance matrix Σ, which will be specified below. The simulation model for case (MA) is generated from an MA(1) process (u_{it}), which can be represented as an infinite order AR process. Using the lag order p_i selected by the modified AIC rule suggested by Ng and Perron (2000, 2001) with the maximum lag order 10[13], we approximate (u_{it}) by an AR(p_i) process as in (9). The approximated autoregression is then estimated by the OLS method.[14]

The AR and MA coefficients, ρ_i's and θ_i's, used in the generation of the errors (u_{it}) are drawn randomly from the uniform distribution. The location and range of the distribution determine the characteristics of the data generating process. The location determines the amount of serial correlation allowed in the individual series, while the range prescribes the degree of heterogeneity allowed in the panel. It turns out that both the location and range of the distribution critically affect the finite sample performances of the tests considered in the paper. For our simulations, we consider the following six cases:

DGP	Cases	Parameters
AR	A1	$\rho_i \sim$ U$(-0.8, 0.8)$
	A2	$\rho_i \sim$ U$(0.2, 0.4)$
	A3	A1 with independent ε_{it}'s
MA	M1	$\theta_i \sim$ U$(-0.8, -0.4)$
	M2	$\theta_i \sim$ U$(-0.4, 0.4)$
	M3	$\theta_i \sim$ U$(0.4, 0.8)$

Cases (A1) and (A2) are considered to allow respectively for wide and narrow ranges for the AR errors. Case (A1) allows the errors to be quite heterogeneous, and thus the t^* statistics designed for homogeneous panels are expected to perform poorly in this case. Case (A2) represents panels with errors which are more homogeneous.[15] Case (A3) is considered to see how the tests perform when individual dynamics are generated by independent innovations. For the models driven by MA errors, we looked at three different cases. Case (M1) is considered to see how the tests behave in the presence of large negative MA roots. Many previous simulations for the univariate model show that the unit root tests yield severe over-rejections in this case. Case (M2) is considered to examine the performances of the tests when mild serial correlations are allowed. We do not expect significant size distortsions in this case. Case (M3) is included to see how the tests perform in the other extreme where the errors have large MA coefficients.[16] All of the above specifications for the serial correlations are used for both classes of models (M) and (T).

The parameter values for the $(N \times N)$ covariance matrix $\Sigma = (\sigma_{ij})$ are also randomly drawn, but with particular attention. To ensure that Σ is a symmetric positive definite matrix and to avoid the near singularity problem, we generate Σ via following steps:

(1) Generate an $(N \times N)$ matrix U from Uniform[0,1].
(2) Construct from U an orthogonal matrix $H = U(U'U)^{-1/2}$.

[13] This procedure was suggested by an Associate Editor to whom I am very grateful. Indeed the use of this selection rule improved our simulation results relative to the previous results appeared in an earlier version of this paper, which are based upon the usual AIC rule.

[14] The Yule-Walker method is not used here due to its finite sample bias problem, which may outweigh the benefits of ensuring stationarity.

[15] Case (A2) is considered mainly to relate to previous simulation studies for panel unit root tests. See Maddala and Wu (1996) and Im, Pesaran and Shin (1997). It seems, however, that the location of distribution in AR models does not affect the performances of the tests significantly.

[16] We also tried the case where the MA coefficient is drawn from the uniform distribution with a wider range, Uniform$(-0.8, 0.8)$. However, the results were not informative since they were mixture of quite drastically different results obtained from different locations chosen randomly from such a wide range covering both extremely large negative and positive MA roots.

(3) Generate a set of N eigen values, $\lambda_1, \ldots, \lambda_N$. Let $\lambda_1 = r > 0$ and $\lambda_N = 1$ and draw $\lambda_2, \ldots, \lambda_{N-1}$ from Uniform$[r,1]$.
(4) Form a diagonal matrix Λ with $(\lambda_1, \ldots, \lambda_N)$ on the diagonal.
(5) Construct the covariance matrix Σ as a spectral representation $\Sigma = H\Lambda H'$.

The covariance matrix constructed this way will surely be symmetric and nonsingular with eigenvalues taking values from r to 1. We set the maximum eigenvalue at 1 since the scale does not matter. The ratio of the minimum eigenvalue to the maximum is therefore determined by the same parameter r. The covariance matrix becomes singular as r tends to zero, and becomes spherical as r approaches to 1. For the simulations, we set r at $r = 0.1$.[17]

For the test of the unit root hypothesis, we set $\alpha_i = 0$ for all $i = 1, \ldots, N$, and investigate the finite sample sizes in relation to the corresponding nominal test sizes. To examine the rejection probabilities of the tests under the alternative of stationarity, we generate α_i's randomly from Uniform$(-0.8, 0)$. The model is thus heterogenous under the alternative. The finite sample performance of the bootstrap tests are compared with that of the t-bar statistic by Im, Pesaran and Shin (1997), which is based on the average of the individual t-statistics computed from the sample ADF regressions (5) with mean and variance modifications. More explicitly, the t-bar statistic is defined as

$$t\text{-bar} = \frac{\sqrt{N}(\bar{t}_N - N^{-1}\sum_{i=1}^N \mathbf{E}(t_i, p_i, \alpha_{i,1}, \ldots, \alpha_{i,p_i}))}{\sqrt{N^{-1}\sum_{i=1}^N \mathrm{var}(t_i, p_i, \alpha_{i,1}, \ldots, \alpha_{i,p_i})}}$$

where t_i is the t-statistic for testing $\alpha_i = 0$ for the i-th sample ADF regression (5), and $\bar{t}_N = N^{-1}\sum_{i=1}^N t_i$. The values of the expectation and variance, $\mathbf{E}(t_i)$ and $\mathrm{var}(t_i)$, for each individual t_i depend on T, the lag order p_i and the coefficients on the lagged differences $\alpha_{i,k}$'s, and are computed via simulations from independent normal samples assuming $\alpha_{i,1} = \cdots = \alpha_{i,p_i} = 0$. Table 2 in Im, Pesaran and Shin (1997) tabulates the values of $\mathbf{E}(t_i)$ and $\mathrm{var}(t_i)$ for $T = 5, 10, 15, 20, 25, 30, 40, 50, 60, 70, 100$ and for $p_i = 1, \ldots, 8$. When the AR order p_i chosen by the selection rule is greater than 8, we replace it by 8 for the construction of the t-bar test since the mean and variance modifications are available only for p_i's upto 8. However, for the construction of our bootstrap tests, we use the original order selected by the rule.

The panels with the cross-sectional dimensions $N = 5, 10$ and the time series dimension $T = 100$ are considered for the 5% size test. Since we are using random parameter values, we simulate 20 times for each case and report the ranges of the finite sample performances of the tests. Each simulation run is carried out with 1,000 simulation iterations, each of which uses bootstrap critical values computed from 500 bootstrap repetitions. The simulation results for the t-bar statistic and our bootstrap tests F^*_{OT}, F^*_{GT}, K^*_{OT}, K^*_{GT}, t^*_{OT} and t^*_{GT} for the models (M) with heterogeneous fixed effects are reported in Tables MS.A1-MP.M3. Tables MS.A1(A2,A3) and MP.A1(A2,A3) report, respectively, the finite sample sizes and powers of the tests for case A1(A2,A3) with the AR errors generated by the DGP defined with the parameters given in A1(A2,A3). Tables MS.M1(M2,M3) and MP.M1(M2,M3) report the finite sample sizes and powers for case M1(M2,M3) with the MA errors generated by the DGP M1(M2,M3). Similarly, Tables TS.A1-TP.M3 report the finite sample sizes and powers of the tests for the models (T) with individual time trends. For each statistic, we report the minimum, mean, median and maximum of the rejection probabilities under the null and under the alternative hypotheses.

In general, the t-bar test suffers from serious size distortions for both models (M) and (T) and for all specifications of serial correlations considered with cross-sectional dependency. The size distortions in the models driven by MA errors or with time trends are much more severe than those

[17] Our bootstrap tests do not seem to depend on the the value of r, but the t-bar statistic does. Though we do not report the details, we observe from a set of simulations that the t-bar tends to have higher rejection probabilities when r is close to 0, relative to the case where Σ is nearly spherical with $r = 0.99$.

in the models driven by AR errors or with fixed effects only.[18] In particular, it suffers from huge upward size distortions for case (M1) with large negative MA roots. As can be seen from Table MS.M1 for the models (M) with heterogeneous fixed effects, the average size of the t-bar tests for the 5% test is 45% for N=5, and increases to 66% for the larger N=10. For cases (M2) and (M3), the t-bar continues to over-reject for both N=5,10, though the magnitude of the distortions is much smaller than in case (M1). See Tables MS.M2 and MS.M3. The t-bar has similar patterns of size distortions for the models (T) with time trends. However, the degree of distortions are noticeably magnified in this case, especially for the cases generated by MA errors. For instance, as can be seen from Table TS.M1, the upward distortion in case (M1) is now enormous, and it gets worse as N increases. For the 5% test, the average size of the t-bar test is 76% for the smaller N=5 and increases to 94% when the larger $N = 10$ is used.

On the other hand, the finite sample sizes of the bootstrap tests are overall quite close to the nominal test sizes in most of the cases. For the cases with AR errors, all bootstrap tests have very good size properties for both classes of models (M) and (T), as can be seen from Tables MS.A1(A2) and TS.A1(A2). Our bootstrap tests also have reasonably good sizes for the cases with MA errors, except for case (M1). In this case, all bootstrap tests also suffer from upward size distortions; however, the degree of the distortions in our bootstrap tests is not comparable to that of the t-bar test. Ours is much less severe than theirs.[19] See Tables MS.M1 and TS.M1.

We now turn to finite sample powers of the tests. In general, it seems that our bootstrap tests perform satisfactorily in all cases we consider in the paper. It is, however, not easy to directly compare power performances of our tests with those of the t-bar using the computed rejection probabilities, since in many cases the t-bar test has significant size distortions. Nonetheless, we can make straightforward comparisons in some cases. As can be seen from Tables MS.A and MP.A for cases A1 and A2 for the models with fixed effects, the GLS-based bootstrap tests, F^*_{GT} and K^*_{GT}, perform clearly better than the t-bar test even in the presence of the upward size distortions in the t-bar test. The bootstrap tests appear to perform quite well relative to the t-bar test also for all other cases, once we take into account the upward size distortions in the t-bar test. Our bootstrap tests F^* and K^* seem to yield better powers than the t-bar test in most of the cases. There are, however, cases where the t^* tests appear less powerful than the t-bar. The performance of the t^* tests varies with the degree of heterogeneity allowed in the model. The finite sample powers of all tests are lower for the models (T) with time trends, compared with the models (M) with fixed effects only. We expect that the efficient GLS-detrending suggested by Elliot, Rothenberg and Stock (1996) improves the power properties of our bootstrap tests in both models. However, it is not used here to give a fair ground to the t-bar test, whose critical values are available only for the usual OLS-detrending.

We now discuss the relative performances among our bootstrap tests. The GLS-based tests generally perform better than their OLS counterparts in terms of both sizes and powers. The GLS-based tests F^*_{GT} and K^*_{GT} have better sizes for all cases except (M1) and are more powerful than their OLS counterparts F^*_{OT} and K^*_{OT}. This is even more evident for the models (T) with individual time trends. For the t^* tests, however, this is not always the case. The t^*_{GT} seems to have better sizes than t^*_{OT} in most of the cases, but it is more powerful than t^*_{OT} only for the smaller N. See the discussion on the t^* test below. Overall, the F^* and K^* tests perform better than the t^* test in terms of both sizes and powers, except for the cases described below.

[18]In the models with independent AR errors as in case A3, the t-bar performs quite well as expected. However, even in such cases with cross-sectional independence, the t-bar test starts to over-reject as we introduce time trends and the upward distortions become more obvious as N increases.

[19]The poor size performances of the tests in case (M1) for both models with fixed effects and time trends go in line with the well known size problems of the univariate unit root tests in models with large negative MA roots. It is also observed that bootstrap tests reduce, though not completely, the size distortions in such cases. See Chang and Park (1999).

The t-type tests are one-sided tests constructed for homogenous panels. Hence, for our simulation models with the alternatives drawn heterogeneously for each individual unit, it is well expected that the t^* tests will be less powerful than the F^* and K^* tests that are designed for heterogeneous panels. Indeed, when the models allow substantial amount of heterogeneity, as in cases (A1) and (A3), the t^* tests have lower power and exhibit larger variability. However, when the models are modestly heterogeneous, as in case (A2), the t^* tests become much less variable and more powerful, almost comparable to the F^* and K^* tests. For the cases with MA errors, the models considered here are not drastically heterogeneous, and consequently the powers of the t^* tests are reasonably good. We also note that the OLS based t-statistic t^*_{OT} is more powerful than its GLS couterpart t^*_{GT} when the larger $N=10$ is used, which is not observed for the F^* and K^* tests.

The K-statistics are proposed as an alternative to the two-sided F-type test to come up with more powerful tests for the unit roots against the one-way alternative of stationarity. The simulation results in Tables MP and TP, however, show that the improvement the K-statistics make over the F-type tests are insignificant. The finite sample distributions of $\hat{\alpha}_{GT}$ and $\hat{\alpha}_{OT}$, upon which the modifications for the K-statistics are made, are indeed skewed to the left so much that the modifications do not have actual effect. For better results, we thus need to correct the biases in the distributions of $\hat{\alpha}_{GT}$ and $\hat{\alpha}_{OT}$ before applying the modifications given in the equation preceding (11). This can be implemented in practice by carrying out a nested bootstrap, the first step of which involves the bootstrap corrections for the biases in $\hat{\alpha}_{GT}$ and $\hat{\alpha}_{OT}$. We do not pursue this in the present paper due to the computation time, but will report in a future work.

5. Conclusion

There has been much recent empirical and theoretical econometric work on models with nonstationary panel data. In particular, much attention has been paid to the development and implementation of the panel unit root tests which have been used frequently to test for various covergence theories, such as growth covergence theories and purchasing power parity hypothesis. A variety of tests have been proposed, including the tests proposed by Levin and Lin (1993) and Im, Pesaran and Shin (1997) that appear to be most commonly used. All the existing tests, however, assume the independence across cross-sectional units, which is quite restrictive for most of economic panel data we encounter. Cross-sectional dependency seems indeed quite apparent for most of interesting panel data.

In the paper, we investigate various unit root tests for panel models which explicitly allow for the cross-correlation across cross-sectional units as well as heterogeneous serial dependence. The limit theories for the panel unit root tests are derived by passing the number of time series observations T to infinity with the number of cross-sectional units N fixed. As expected the limit distributions of the tests are nonstandard and depend heavily on the nuisance parameters, rendering the standard inferential procedure invalid. To overcome the inferential difficulty of the panel unit root tests in the presence of cross-sectional dependency, we propose to use the bootstrap method. Limit theories for the bootstrap tests are developed, and in particular their asymptotic validity is established by proving the consistency of the boostrap tests. The simulations show that the bootstrap panel unit root tests perform well in finite samples relative to the t-bar statistic by Im, Pesaran and Shin (1997).

6. Appendix: Mathematical Proofs

The following lemmas provide asymptotic results for the sample moments appearing in the sample test statistics F_{GT}, F_{OT}, K_{GT}, K_{OT}, t_{GT} and t_{OT} defined in (7), (8), (11), (12) and (14).

Lemma A1 Under Assumptions 1 – 3, we have

(a) $\displaystyle \frac{1}{T}\sum_{t=1}^{N} y_{i,t-1}\varepsilon_{jt}^{p_j} = \pi_i(1)\frac{1}{T}\sum_{t=1}^{T} w_{i,t-1}\varepsilon_{jt} + o_p(1)$, for all $i,j = 1,\ldots,N$

(b) $\displaystyle \frac{1}{T^2}\sum_{t=1}^{T} y_{i,t-1}y_{j,t-1} = \pi_i(1)\pi_j(1)\frac{1}{T^2}\sum_{t=1}^{T} w_{i,t-1}w_{j,t-1} + o_p(1)$, for all $i,j = 1,\ldots,N$

(c) $\displaystyle \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^p\varepsilon_t^{p\prime} = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\varepsilon_t' + o_p(1)$

Proof of Lemma A1

Part (a) The stated results follow immediately if we apply the results in Lemma 3.1 (a) of Chang and Park (2001) to each (i,j) pair, for $i,j = 1,\ldots,N$.

Part (b) The stated result follows directly from Phillips and Solo (1992).

Part (c) Let $Q_T = T^{-1}\sum_{t=1}^{T}\varepsilon_t^p\varepsilon_t^{p\prime} - T^{-1}\sum_{t=1}^{T}\varepsilon_t\varepsilon_t'$. Then for each (i,j)-element of Q, we have

$$Q_{T,ij} = \frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{it}^{p_i} - \varepsilon_{it})\varepsilon_{jt}^{p_j} + \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}(\varepsilon_{jt}^{p_j} - \varepsilon_{jt}) = o_p(p_i^{-s}) + o_p(p_j^{-s})$$

due to Lemma 3.1 (c) in Chang and Park (2001). Now the stated result is immediate.

Lemma A2 Under Assumptions 1 – 3, we have

(a) $\displaystyle \left\| \left(\frac{1}{T}\sum_{t=1}^{T} x_{it}^{p_i}x_{it}^{p_i\prime}\right)^{-1}\right\| = O_p(1)$, for all p_i and $i = 1,\ldots,N$

(b) $\displaystyle \left|\sum_{t=1}^{T} x_{it}^{p_i}y_{j,t-1}\right| = O_p(Tp_i^{1/2})$, for all $i,j = 1,\ldots,N$

(c) $\displaystyle \left|\sum_{t=1}^{T} x_{it}^{p_i}\varepsilon_{jt}^{p_j}\right| = O_p(T^{1/2}p_i^{1/2}) + o_p(Tp_i^{1/2}p_j^{-s})$, for all $i,j = 1,\ldots,N$.

Proof of Lemma A2 The stated result in Part (a) follows directly from the application of the result in Lemma 3.2 (a) of Chang and Park (2001) for each $i = 1,\ldots,N$, and those in Parts (b) and (c) are easily obtained using the results in Lemma 3.2 (b) and (c) of the aforementioned reference for each (i,j) pair for $i,j = 1,\ldots,N$, with some obvious modifications with respect to the heterogeneous orders p_i's of the AR approximations involved.

Proof of Theorem A.1

Part (a) We begin by examining the stochastic orders of the component sample moments appearing in A_{GT} and B_{GT} defined below (8). Let $\lambda(\cdot)$ denote eigenvalues of a matrix. We have

$$\lambda_{\min}(\tilde{\Sigma}^{-1}\otimes I_T)X_p'X_p \leq X_p'(\tilde{\Sigma}^{-1}\otimes I_T)X_p$$

Notice that $\lambda_{\min}(\tilde{\Sigma}^{-1}\otimes I_T) = \lambda_{\min}(\tilde{\Sigma}^{-1})$ and $\lambda_{\min}(\tilde{\Sigma}^{-1}) = 1/\lambda_{\max}(\tilde{\Sigma})$. Then we have

$$\left(\frac{X_p'(\tilde{\Sigma}^{-1}\otimes I_T)X_p}{T}\right)^{-1} \leq \lambda_{\max}(\tilde{\Sigma})\left(\frac{X_p'X_p}{T}\right)^{-1} = O_p(1) \tag{28}$$

since $\lambda_{\max}(\tilde{\Sigma}) \to_p \lambda_{\max}(\Sigma) < \infty$ and $(T^{-1}X_p'X_p)^{-1} = O_p(1)$ due to Lemma A2 (a). Moreover it follows from Lemma A2 (b) that

$$X_p'(\tilde{\Sigma}^{-1} \otimes I_T)Y_\ell = O_p(T\bar{p}^{1/2}) \tag{29}$$

where $\bar{p} = \max_{1 \le i \le N} p_i$, and from Lemma A2 (c) that

$$X_p'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p = O_p(T^{1/2}\bar{p}^{1/2}) + o_p(T\bar{p}^{1/2}\underline{p}^{-s}) \tag{30}$$

where $\underline{p} = \min_{1 \le i \le N} p_i$. Notice that $\bar{p} = \underline{p} = o(T^{1/2})$ as $T \to \infty$ under Assumption 3.

It follows from (28), (29) and (30) that

$$\begin{aligned}
&\left| Y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \left(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \right)^{-1} X_p'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p \right| \\
&\le \left| Y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \right| \left\| \left(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \right)^{-1} \right\| \left| X_p'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p \right| \\
&= o_p(T\bar{p}\underline{p}^{-s}) + O_p(T^{1/2}\bar{p})
\end{aligned}$$

which implies

$$\frac{A_{GT}}{T} = \frac{Y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p}{T} + o_p(1) = Q_{A_{GT}} + o_p(1) \tag{31}$$

due to Lemma A1 (a), where

$$Q_{A_{GT}} = \begin{pmatrix} \sum_{j=1}^{N} \tilde{\sigma}^{1j}\pi_1(1)\frac{1}{T}\sum_{t=1}^{T} w_{1,t-1}\varepsilon_{jt} \\ \vdots \\ \sum_{j=1}^{N} \tilde{\sigma}^{Nj}\pi_N(1)\frac{1}{T}\sum_{t=1}^{T} w_{N,t-1}\varepsilon_{jt} \end{pmatrix}$$

where $\tilde{\sigma}_{ij}$ denotes (i,j)-element of the covariance matrix estimate $\tilde{\Sigma}$.

Moreover, we have from (28) and (29) that

$$\begin{aligned}
&\left| Y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \left(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \right)^{-1} X_p'(\tilde{\Sigma}^{-1} \otimes I_T)Y_\ell \right| \\
&\le \left| Y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \right| \left\| \left(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \right)^{-1} \right\| \left| X_p'(\tilde{\Sigma}^{-1} \otimes I_T)Y_\ell \right| = O_p(T\bar{p})
\end{aligned}$$

which, together with Lemma A1 (b) gives

$$\frac{B_{GT}}{T^2} = \frac{Y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)Y_\ell}{T^2} + o_p(1) = Q_{B_{GT}} + o_p(1) \tag{32}$$

where

$$Q_{B_{GT}} = \begin{pmatrix} \tilde{\sigma}^{11}\pi_1(1)^2\frac{1}{T^2}\sum_{t=1}^{T} w_{1,t-1}^2 & \cdots & \tilde{\sigma}^{1N}\pi_1(1)\pi_N(1)\frac{1}{T^2}\sum_{t=1}^{T} w_{1,t-1}w_{N,t-1} \\ \vdots & \vdots & \vdots \\ \tilde{\sigma}^{N1}\pi_N(1)\pi_1(1)\frac{1}{T^2}\sum_{t=1}^{T} w_{N,t-1}w_{1,t-1} & \cdots & \tilde{\sigma}^{NN}\pi_N(1)^2\frac{1}{T^2}\sum_{t=1}^{T} w_{N,t-1}^2 \end{pmatrix}$$

Using the asymptotic results in (31) and (32), we write

$$F_{GT} = \left(\frac{A_{GT}}{T}\right)' \left(\frac{B_{GT}}{T^2}\right)^{-1} \left(\frac{A_{GT}}{T}\right) = Q'_{A_{GT}} Q^{-1}_{B_{GT}} Q_{A_{GT}} + o_p(1)$$

Then the limit distribution of F_{GT} follows immediately from the invariance principle given in (25).

Part (b) We have from Lemma A2 (b) and (c) that

$$X_p' Y_\ell = O_p(T\bar{p}^{1/2}), \quad X_p'\varepsilon_p = O_p(T^{1/2}\bar{p}^{1/2}) + o_p(T\bar{p}^{1/2}\underline{p}^{-s}) \tag{33}$$

These together with (28) give

$$\left|Y_\ell' X_p (X_p'X_p)^{-1} X_p'\varepsilon_p\right| \;\leq\; |Y_\ell' X_p| \left\|(X_p'X_p)^{-1}\right\| \left|X_p'\varepsilon_p\right| \;=\; o_p(T\bar{p}\underline{p}^{-s}) + O_p(T^{1/2}\bar{p})$$

which in turn gives

$$\frac{A_{OT}}{T} = \frac{Y_\ell'\varepsilon_p}{T} + o_p(1) = Q_{A_{OT}} + o_p(1) \tag{34}$$

due to Lemma A1 (a), where

$$Q_{A_{OT}} = \begin{pmatrix} \pi_1(1)\dfrac{1}{T}\displaystyle\sum_{t=1}^{T} w_{1,t-1}\varepsilon_{1t} \\ \vdots \\ \pi_N(1)\dfrac{1}{T}\displaystyle\sum_{t=1}^{T} w_{N,t-1}\varepsilon_{Nt} \end{pmatrix}$$

We have from (28) that

$$X_p'(\tilde{\Sigma}\otimes I_T)X_p \leq \lambda_{\max(\tilde{\Sigma})}(X_p'X_p) = O_p(T) \tag{35}$$

Also it follows from Lemma A2 (b) that $X_p'(\tilde{\Sigma}\otimes I_T)Y_\ell = O_p(T\bar{p}^{1/2})$.

Then we have $\left|Y_\ell' X_p (X_p'X_p)^{-1} X_p'(\tilde{\Sigma}\otimes I_T)Y_\ell\right| = O_p(T\bar{p})$ and

$$\left|Y_\ell' X_p (X_p'X_p)^{-1} X_p'(\tilde{\Sigma}\otimes I_T)X_p(X_p'X_p)^{-1}X_p'Y_\ell\right| = O_p(T\bar{p})$$

which in turn give

$$\frac{M_{FOT}}{T^2} = \frac{Y_\ell'(\tilde{\Sigma}\otimes I_T)Y_\ell}{T^2} + o_p(1) = Q_{M_{FOT}} + o_p(1) \tag{36}$$

due to Lemma A1 (b), where

$$Q_{M_{FOT}} = \begin{pmatrix} \tilde{\sigma}_{11}\pi_1(1)^2\dfrac{1}{T^2}\displaystyle\sum_{t=1}^{T} w_{1,t-1}^2 & \cdots & \tilde{\sigma}_{1N}\pi_1(1)\pi_N(1)\dfrac{1}{T^2}\displaystyle\sum_{t=1}^{T} w_{1,t-1}w_{N,t-1} \\ \vdots & \vdots & \vdots \\ \tilde{\sigma}_{N1}\pi_N(1)\pi_1(1)\dfrac{1}{T^2}\displaystyle\sum_{t=1}^{T} w_{N,t-1}w_{1,t-1} & \cdots & \tilde{\sigma}_{NN}\pi_N(1)^2\dfrac{1}{T^2}\displaystyle\sum_{t=1}^{T} w_{N,t-1}^2 \end{pmatrix}$$

We now have from the results in (34) and (36) that

$$F_{OT} = \left(\frac{A_{OT}}{T}\right)' \left(\frac{M_{FOT}}{T^2}\right)^{-1} \left(\frac{A_{OT}}{T}\right) = Q'_{A_{OT}} Q^{-1}_{M_{FOT}} Q_{A_{OT}} + o_p(1)$$

from which the stated result follows immediately.

Proof of Corollary A.1

Part (a) It follows from (31) and (32) that

$$T\hat{\alpha}_{GT} = \left(\frac{B_{GT}}{T^2}\right)^{-1}\left(\frac{A_{GT}}{T}\right) = Q_{B_{GT}}^{-1} Q_{A_{GT}} + o_p(1)$$

which implies

$$\frac{1}{T}\Big(A_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\}\Big) = \left(\frac{A_{GT}}{T} .* 1\left\{\frac{\hat{\alpha}_{GT}}{T} \leq 0\right\}\right) = \left(Q_{A_{GT}} .* 1\left\{Q_{B_{GT}}^{-1} Q_{A_{GT}} \leq 0\right\}\right) + o_p(1)$$

Due to the above result and (32), we may write the K_{GT} statistic given in (11) as

$$\begin{aligned} K_{GT} &= \left(\frac{1}{T}\Big(A_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\}\Big)\right)' \left(\frac{B_{GT}}{T^2}\right)^{-1} \left(\frac{1}{T}\Big(A_{GT} .* 1\{\hat{\alpha}_{GT} \leq 0\}\Big)\right) \\ &= \left(Q_{A_{GT}} .* 1\left\{Q_{B_{GT}}^{-1} Q_{A_{GT}} \leq 0\right\}\right)' Q_{B_{GT}}^{-1} \left(Q_{A_{GT}} .* 1\left\{Q_{B_{GT}}^{-1} Q_{A_{GT}} \leq 0\right\}\right) + o_p(1) \end{aligned}$$

Now the stated result follows immediately from (25).

Part (b) From (28) and (33), we have $\left|Y_\ell' X_p (X_p' X_p)^{-1} X_p' Y_\ell\right| = O_p(T\bar{p})$, which together with Lemma A1 (b) gives

$$\frac{B_{OT}}{T^2} = \frac{Y_\ell' Y_\ell}{T^2} + o_p(1) = Q_{B_{OT}} + o_p(1)$$

where

$$Q_{B_{OT}} = \begin{pmatrix} \pi_1(1)^2 \frac{1}{T^2}\sum_{t=1}^{T} w_{1,t-1}^2 & \cdots & \pi_1(1)\pi_N(1)\frac{1}{T^2}\sum_{t=1}^{T} w_{1,t-1}w_{N,t-1} \\ \vdots & \vdots & \vdots \\ \pi_N(1)\pi_1(1)\frac{1}{T^2}\sum_{t=1}^{T} w_{N,t-1}w_{1,t-1} & \cdots & \pi_N(1)^2\frac{1}{T^2}\sum_{t=1}^{T} w_{N,t-1}^2 \end{pmatrix} \tag{37}$$

It follows from (34) and the above result that

$$T\hat{\alpha}_{OT} = \left(\frac{B_{OT}}{T^2}\right)^{-1}\left(\frac{A_{OT}}{T}\right) = Q_{B_{OT}}^{-1} Q_{A_{OT}} + o_p(1)$$

and

$$\frac{1}{T}\Big(A_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\}\Big) = \left(Q_{A_{OT}} .* 1\left\{Q_{B_{OT}}^{-1} Q_{A_{OT}} \leq 0\right\}\right) + o_p(1)$$

From this and the result in (36), we may express the statistic K_{OT} given in (12) as

$$\begin{aligned} K_{OT} &= \left(\frac{1}{T}\Big(A_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\}\Big)\right)' \left(\frac{M_{FOT}}{T^2}\right)^{-1} \left(\frac{1}{T}\Big(A_{OT} .* 1\{\hat{\alpha}_{OT} \leq 0\}\Big)\right) \\ &= \left(Q_{A_{OT}} .* 1\left\{Q_{B_{OT}}^{-1} Q_{A_{OT}} \leq 0\right\}\right)' Q_{M_{FOT}}^{-1} \left(Q_{A_{OT}} .* 1\left\{Q_{B_{OT}}^{-1} Q_{A_{OT}} \leq 0\right\}\right) + o_p(1) \end{aligned}$$

which is required for the stated result.

Proof of Theorem A.2 The limit theories for the GLS and OLS based t-statistics t_{GT} and t_{OT} defined in (14) can be derived in the similar manner as we did for the F-type tests F_{GT} and F_{OT} in the proof of Theorem A.1. We just have to take into account that the lagged level variables come in as an $(NT \times 1)$-vector y_ℓ instead of the $(NT \times N)$-matrix Y_ℓ.

Part (a) We begin by examining the sample moments appearing in a_{GT} and b_{GT}, defined below (14). Since $X_p'(\tilde{\Sigma}^{-1} \otimes I_T)y_\ell = O_p(T\bar{p}^{1/2})$ due to Lemma A2 (b), it follows from (28) and (30) that

$$\left| y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \left(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p\right)^{-1} X_p'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p \right| = o_p(T\bar{p}\underline{p}^{-s}) + O_p(T^{1/2}\bar{p})$$

and

$$\left| y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)X_p \left(X_p'(\tilde{\Sigma}^{-1} \otimes I_T)X_p\right)^{-1} X_p'(\tilde{\Sigma}^{-1} \otimes I_T)y_\ell \right| = O_p(T\bar{p})$$

Then from the above results and Lemma A1 (a) and (b), it follows that

$$\begin{aligned} \frac{a_{GT}}{T} &= \frac{y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon_p}{T} + o_p(1) \;=\; \sum_{i=1}^N \sum_{j=1}^N \tilde{\sigma}^{ij} \frac{1}{T} \sum_{t=1}^T y_{i,t-1}\varepsilon_{jt}^{p_j} + o_p(1) \;=\; Q_{a_{GT}} + o_p(1) \\ \frac{b_{GT}}{T^2} &= \frac{y_\ell'(\tilde{\Sigma}^{-1} \otimes I_T)y_\ell}{T^2} + o_p(1) \;=\; \sum_{i=1}^N \sum_{j=1}^N \tilde{\sigma}^{ij} \frac{1}{T^2} \sum_{t=1}^T y_{i,t-1}y_{j,t-1} + o_p(1) \;=\; Q_{b_{GT}} + o_p(1) \end{aligned}$$

where

$$\begin{aligned} Q_{a_{GT}} &= \sum_{i=1}^N \sum_{j=1}^N \tilde{\sigma}^{ij} \pi_i(1) \frac{1}{T} \sum_{t=1}^T w_{i,t-1}\varepsilon_{jt} \\ Q_{b_{GT}} &= \sum_{i=1}^N \sum_{j=1}^N \tilde{\sigma}^{ij} \pi_i(1)\pi_j(1) \frac{1}{T^2} \sum_{t=1}^T w_{i,t-1}w_{j,t-1} \end{aligned}$$

We may now write t_{GT} defined in (14) as follows

$$t_{GT} = \frac{a_{GT}}{T}\left(\frac{b_{GT}}{T^2}\right)^{-1/2} = Q_{a_{GT}}Q_{b_{GT}}^{-1/2} + o_p(1)$$

and the limit theory for t_{GT} is directly obtained from applying the invariance principle in (25) to $Q_{a_{GT}}$ and $Q_{b_{GT}}$.

Part (b) Again, we first analyze the components a_{OT} and M_{tOT}, defined below (14), that constitute the OLS based t-statistic t_{OT} given in (14). Since

$$X_p'y_\ell = O_p(T\bar{p}^{1/2}) \quad \text{and} \quad X_p'(\tilde{\Sigma} \otimes I_T)y_\ell = O_p(T\bar{p}^{1/2})$$

by Lemma A2 (b), we have from (35) that

$$\begin{aligned} \left|Y_\ell'X_p(X_p'X_p)^{-1}X_p'\varepsilon_p\right| &= o_p(T\bar{p}\underline{p}^{-s}) + O_p(T^{1/2}\bar{p}) \\ \left|Y_\ell'X_p(X_p'X_p)^{-1}X_p'(\tilde{\Sigma} \otimes I_T)Y_\ell\right| &= O_p(T\bar{p}) \\ \left|Y_\ell'X_p(X_p'X_p)^{-1}X_p'(\tilde{\Sigma} \otimes I_T)X_p(X_p'X_p)^{-1}X_p'Y_\ell\right| &= O_p(T\bar{p}) \end{aligned}$$

We now deduce from Lemma A1 (a) and (b) that

$$\begin{aligned}
\frac{a_{OT}}{T} &= \frac{y_\ell' \varepsilon_p}{T} + o_p(1) = \sum_{i=1}^N \frac{1}{T}\sum_{t=1}^T y_{i,t-1}\varepsilon_{it}^{p_i} + o_p(1) = Q_{a_{OT}} + o_p(1) \\
\frac{M_{tOT}}{T^2} &= \frac{y_\ell'(\tilde\Sigma \otimes I_T) y_\ell}{T^2} + o_p(1) = \sum_{i=1}^N\sum_{j=1}^N \tilde\sigma_{ij}\frac{1}{T^2}\sum_{t=1}^T y_{i,t-1}y_{j,t-1} + o_p(1) = Q_{M_{tOT}} + o_p(1)
\end{aligned}$$

where

$$\begin{aligned}
Q_{a_{OT}} &= \sum_{i=1}^N \pi_i(1)\frac{1}{T}\sum_{t=1}^T w_{i,t-1}\varepsilon_{it} \\
Q_{M_{tOT}} &= \sum_{i=1}^N\sum_{j=1}^N \tilde\sigma_{ij}\pi_i(1)\pi_j(1)\frac{1}{T^2}\sum_{t=1}^T w_{i,t-1}w_{j,t-1}
\end{aligned}$$

Then we have

$$t_{OT} = \frac{a_{OT}}{T}\left(\frac{M_{tOT}}{T^2}\right)^{-1/2} = Q_{a_{OT}}Q_{M_{tOT}}^{-1/2} + o_p(1)$$

from which the stated result follows immediately.

Proofs for the Bootstrap Asymptotics

In the following lemma, we use an operator norm for matrices: if $C = (c_{ij})$ is a matrix, then we let $\|C\| = \max_x |Cx|/|x|$.

Lemma B2 Let $x_{it}^{*p_i} = (\triangle y_{i,t-1}^*, \ldots, \triangle y_{i,t-p_i}^*)'$. Then we have

(a) $\mathbf{E}^* \left\| \left(\frac{1}{T}\sum_{t=1}^T x_{it}^{*p_i} x_{it}^{*p_i\prime} \right)^{-1} \right\| = O_p(1)$, for all $i = 1, \ldots, N$.

(b) $\mathbf{E}^* \left| \sum_{t=1}^T x_{it}^{*p_i} y_{j,t-1}^* \right| = O(Tp_i^{1/2})$ a.s., for all $i, j = 1, \ldots, N$.

(c) $\mathbf{E}^* \left| \sum_{t=1}^T x_{it}^{*p_i} \varepsilon_{jt}^* \right| = O(T^{1/2}p_i^{1/2})$ a.s., for all $i, j = 1, \ldots, N$.

under Assumptions 1 – 3.

Proofs of Lemmas B1 and B2 The stated results follow directly from Lemmas 3.2 and 3.3 of Chang and Park (1999), and thus omitted.

Proof of Theorem B.1 The proof here follows closely the lines of the proof of Theorem A.1, using the bootstrap asymptotics established in Lemmas 1 and 2.

Part (a) From Lemma 2 (a), we have

$$\left(\frac{X_p^{*\prime}(\tilde\Sigma^{-1}\otimes I_T)X_p^*}{T}\right)^{-1} \leq \lambda_{\max}(\tilde\Sigma)\left(\frac{X_p^{*\prime}X_p^*}{T}\right)^{-1} = O_p^*(1) \tag{38}$$

which along with the results in Lemma B1 (b) and Lemma B2 (b) and (c) gives

$$\frac{A_{GT}^*}{T} = Y_\ell^{*\prime}(\tilde\Sigma^{-1}\otimes I_T)\varepsilon^* + o_p^*(1) = Q_{A_{GT}^*} + o_p^*(1) \tag{39}$$

in **P** under Assumptions 1 – 3, where $Q_{A^*_{GT}}$ is defined similarly as $Q_{A_{GT}}$ in (31) with $\tilde{\pi}_i(1)$, w^*_{it} and ε^*_{it} in the places of $\pi_i(1)$, w_{it} and ε_{it}. Similarly, we have from (38), Lemma B1 (b) and Lemma 2 (b) that

$$\frac{B^*_{GT}}{T^2} = Y^{*\prime}_\ell(\tilde{\Sigma}^{-1} \otimes I_T)Y^*_\ell + o^*_p(1) = Q_{B^*_{GT}} + o^*_p(1) \tag{40}$$

in **P** under Assumptions 1 – 3, analogously as before, where $Q_{B^*_{GT}}$ is defined similarly as $Q_{B_{GT}}$ given below (32) with $\tilde{\pi}_i(1)$ and w^*_{it} in the places of $\pi_i(1)$ and w_{it}, respectively.

We now write the bootstrapped statistic F^*_{GT} as

$$F^*_{GT} = \left(\frac{A^*_{GT}}{T}\right)' \left(\frac{B^*_{GT}}{T^2}\right)^{-1} \left(\frac{A^*_{GT}}{T}\right) = Q'_{A^*_{GT}} Q^{-1}_{B^*_{GT}} Q_{A^*_{GT}} + o^*_p(1)$$

due to (39) and (40). It is shown in Park (1999) that

$$\tilde{\pi}_i(1) \to_{a.s.} \pi_i(1) \tag{41}$$

and, using the multivariate bootstrap invariance principle developed in Chang, Park and Song (2000), we have

$$\frac{1}{T}\sum_{t=1}^{T} w^*_{t-1}\varepsilon^{*\prime}_t \to_{d^*} \int_0^1 BdB' \text{ a.s.}, \quad \frac{1}{T^2}\sum_{t=1}^{T} w^*_{t-1}w^{*\prime}_{t-1} \to_{d^*} \int_0^1 BB' \text{ a.s.} \tag{42}$$

under Assumptions 1 – 3. Now, the limiting distribution of the F^*_{GT} follows immediately.

Part (b) It follows from Lemma 2 (b) and (c) that

$$X^{*\prime}_p Y^*_\ell = O^*_p(T\bar{p}^{1/2}), \quad X^{*\prime}_p \varepsilon^* = O^*_p(T^{1/2}\bar{p}^{1/2}) \tag{43}$$

which together with (38) and Lemma B1 (a) implies that

$$\frac{A^*_{OT}}{T} = \frac{Y^{*\prime}_\ell \varepsilon^*}{T} + o^*_p(1) = Q_{A^*_{OT}} + o^*_p(1) \tag{44}$$

where $Q_{A^*_{OT}}$ is defined similarly as $Q_{A_{OT}}$ in (34) with the bootstrap samples and $\tilde{\pi}(1)$.

Next, we deduce from (38) and Lemma 2 (b) that

$$X^{*\prime}_p(\tilde{\Sigma} \otimes I_T)X^*_p = O^*_p(T^{-1}), \quad X^{*\prime}_p(\tilde{\Sigma} \otimes I_T)Y^*_\ell = O^*_p(T\bar{p}^{1/2}) \tag{45}$$

and this together with (43) gives

$$\frac{M^*_{FOT}}{T^2} = \frac{Y^{*\prime}_\ell(\tilde{\Sigma} \otimes I_T)Y^*_\ell}{T^2} + o^*_p(1) = Q_{M^*_{FOT}} + o^*_p(1) \tag{46}$$

due to Lemma B1 (b), where $Q_{M^*_{FOT}}$ is the bootstrap counterpart of $Q_{M_{FOT}}$ given in (36).

Finally, we have from the results in (45) and (46)

$$F^*_{OT} = \left(\frac{A^*_{OT}}{T}\right)' \left(\frac{M^*_{FOT}}{T^2}\right)^{-1} \left(\frac{A^*_{OT}}{T}\right) = Q'_{A^*_{OT}} Q^{-1}_{M^*_{FOT}} Q_{A^*_{OT}} + o^*_p(1)$$

and the stated result now follows immediately from (41) and (42).

Proof of Corollary B.1 The proof is analogous to the proof of Corollary A.1.

Part (a) We have from the bootstrap asymptotic results established in (39) and (40) that

$$T\hat{\alpha}^*_{GT} = \left(\frac{B^*_{GT}}{T^2}\right)^{-1}\left(\frac{A^*_{GT}}{T}\right) = Q^{-1}_{B^*_{GT}} Q_{A^*_{GT}} + o^*_p(1)$$

Then we may write the K^*_{GT} statistic given in (22) as

$$\begin{aligned} K^*_{GT} &= \left(\frac{1}{T}\left(A^*_{GT} .* 1\{\hat{\alpha}^*_{GT} \le 0\}\right)\right)'\left(\frac{B^*_{GT}}{T^2}\right)^{-1}\left(\frac{1}{T}\left(A^*_{GT} .* 1\{\hat{\alpha}^*_{GT} \le 0\}\right)\right) \\ &= \left(Q_{A^*_{GT}} .* 1\left\{Q^{-1}_{B^*_{GT}} Q_{A^*_{GT}} \le 0\right\}\right)' Q^{-1}_{B^*_{GT}}\left(Q_{A^*_{GT}} .* 1\left\{Q^{-1}_{B^*_{GT}} Q_{A^*_{GT}} \le 0\right\}\right) + o^*_p(1) \end{aligned}$$

Now the stated result follows immediately from (41) and (42).

Part (b) Using the bootstrap asymptotic results in (38), (43) and Lemma B1 (b), we derive

$$\frac{B^*_{OT}}{T^2} = \frac{Y^{*\prime}_\ell Y^*_\ell}{T^2} + o_p(1) = Q_{B^*_{OT}} + o_p(1)$$

where $Q_{B^*_{OT}}$ is the bootstrap counterpart of $Q_{B_{OT}}$ defined in (37). Similarly as in the proof of Corollary A.1 (b), we may write the test K^*_{OT} given in (22) as

$$\begin{aligned} K^*_{OT} &= \left(\frac{1}{T}\left(A^*_{OT} .* 1\{\hat{\alpha}^*_{OT} \le 0\}\right)\right)'\left(\frac{M^*_{FOT}}{T^2}\right)^{-1}\left(\frac{1}{T}\left(A^*_{OT} .* 1\{\hat{\alpha}^*_{OT} \le 0\}\right)\right) \\ &= \left(Q_{A^*_{OT}} .* 1\left\{Q^{-1}_{B^*_{OT}} Q_{A^*_{OT}} \le 0\right\}\right)' Q^{-1}_{M^*_{FOT}}\left(Q_{A^*_{OT}} .* 1\left\{Q^{-1}_{B^*_{OT}} Q_{A^*_{OT}} \le 0\right\}\right) + o^*_p(1) \end{aligned}$$

using the result in (46). The stated result now follows immediately from (41) and (42).

Proof of Theorem B.2 The limit distributions of the bootstrap GLS and OLS based t-statistics, t^*_{GT} and t^*_{OT}, defined in (24) are derived analogously as we did for the sample t-statistics t_{GT} and t_{OT} in the proof of Theorem A.2, using the bootstrap asymptotics established in Lemmas 1 and 2.

Part (a) It follows from Lemmas 1 and 2, and the result in (38) that

$$\begin{aligned} \frac{a^*_{GT}}{T} &= \frac{y^{*\prime}_\ell(\tilde{\Sigma}^{-1} \otimes I_T)\varepsilon^*}{T} + o^*_p(1) = Q_{a^*_{GT}} + o^*_p(1) \\ \frac{b^*_{GT}}{T^2} &= \frac{y^{*\prime}_\ell(\tilde{\Sigma}^{-1} \otimes I_T)y^*_\ell}{T^2} + o^*_p(1) = Q_{b^*_{GT}} + o^*_p(1) \end{aligned}$$

where $Q_{a^*_{GT}}$ and $Q_{b^*_{GT}}$ are the bootstrap counterparts of $Q_{a_{GT}}$ and $Q_{b_{GT}}$ defined in the proof of Theorem A.2 (a). We may now write t^*_{GT} as

$$t^*_{GT} = \frac{a^*_{GT}}{T}\left(\frac{b^*_{GT}}{T^2}\right)^{-1/2} = Q_{a^*_{GT}} Q^{-1/2}_{b^*_{GT}} + o^*_p(1)$$

and the limit theory for t^*_{GT} is directly obtained from (41) and (42).

Part (b) We have

$$X^{*\prime}_p y^*_\ell = O^*_p(T\bar{p}^{1/2}), \quad X^{*\prime}_p(\tilde{\Sigma} \otimes I_T)y^*_\ell = O^*_p(T\bar{p}^{1/2})$$

by Lemma A2 (b). Then we may deduce from Lemma B1 and (45) that

$$\begin{aligned} \frac{a^*_{OT}}{T} &= \frac{y^{*\prime}_\ell \varepsilon^*}{T} + o^*_p(1) = Q_{a^*_{OT}} + o^*_p(1) \\ \frac{M^*_{tOT}}{T^2} &= \frac{y^{*\prime}_\ell(\tilde{\Sigma} \otimes I_T)y^*_\ell}{T^2} + o^*_p(1) = Q_{M^*_{tOT}} + o^*_p(1) \end{aligned}$$

where $Q_{a^*_{OT}}$ and $Q_{M^*_{tOT}}$ are the bootstrap counterparts to $Q_{a_{OT}}$ and $Q_{M_{tOT}}$ given in the proof of Theorem A.2 (b). Then we have

$$t^*_{OT} = \frac{a^*_{OT}}{T}\left(\frac{M^*_{tOT}}{T^2}\right)^{-1/2} = Q_{a^*_{OT}}Q^{-1/2}_{M^*_{tOT}} + o^*_p(1)$$

The stated result now follows immediately from (41) and (42).

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Table MS.A: Finite Sample Sizes for AR Errors, Demeaned Case

			(A1) ρ_i ~U$(-0.8, 0.8)$				(A2) ρ_i ~U$(0.2, 0.4)$				(A3) ρ_i ~U$(-0.8, 0.8)$, ID			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.052	0.069	0.070	0.082	0.055	0.070	0.070	0.085	0.040	0.053	0.052	0.074
		F^*_{GT}	0.042	0.054	0.054	0.072	0.042	0.054	0.052	0.070	0.036	0.051	0.051	0.065
		F^*_{OT}	0.041	0.054	0.055	0.067	0.043	0.054	0.054	0.069	0.040	0.055	0.055	0.064
		K^*_{GT}	0.041	0.054	0.055	0.074	0.041	0.053	0.052	0.071	0.036	0.051	0.051	0.065
		K^*_{OT}	0.040	0.055	0.055	0.067	0.043	0.054	0.055	0.068	0.040	0.055	0.055	0.065
		t^*_{GT}	0.037	0.050	0.050	0.061	0.039	0.049	0.049	0.064	0.037	0.051	0.053	0.062
		t^*_{OT}	0.036	0.051	0.050	0.066	0.040	0.053	0.054	0.068	0.039	0.054	0.055	0.069
10	100	t-bar	0.061	0.073	0.074	0.086	0.054	0.072	0.072	0.084	0.046	0.056	0.056	0.070
		F^*_{GT}	0.044	0.054	0.054	0.064	0.037	0.054	0.053	0.068	0.045	0.053	0.051	0.064
		F^*_{OT}	0.042	0.058	0.059	0.070	0.047	0.057	0.057	0.066	0.050	0.061	0.061	0.074
		K^*_{GT}	0.044	0.055	0.055	0.065	0.036	0.054	0.054	0.069	0.044	0.054	0.052	0.065
		K^*_{OT}	0.045	0.058	0.058	0.070	0.045	0.057	0.057	0.068	0.050	0.061	0.061	0.074
		t^*_{GT}	0.037	0.053	0.053	0.071	0.048	0.057	0.058	0.069	0.039	0.058	0.058	0.075
		t^*_{OT}	0.039	0.056	0.057	0.071	0.043	0.059	0.060	0.074	0.044	0.064	0.067	0.078

Table MP.A: Finite Sample Powers for AR Errors, Demeaned Case

			(A1) ρ_i ~U$(-0.8, 0.8)$				(A2) ρ_i ~U$(0.2, 0.4)$				(A3) ρ_i ~U$(-0.8, 0.8)$, ID			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.549	0.856	0.885	0.992	0.522	0.842	0.901	0.989	0.568	0.873	0.913	0.997
		F^*_{GT}	0.479	0.863	0.929	0.998	0.447	0.844	0.907	0.992	0.457	0.833	0.888	0.993
		F^*_{OT}	0.354	0.726	0.723	0.952	0.326	0.695	0.701	0.928	0.462	0.842	0.900	0.997
		K^*_{GT}	0.480	0.863	0.929	0.998	0.448	0.845	0.909	0.993	0.462	0.834	0.889	0.993
		K^*_{OT}	0.357	0.728	0.724	0.952	0.326	0.697	0.702	0.930	0.464	0.843	0.901	0.997
		t^*_{GT}	0.066	0.558	0.562	0.994	0.346	0.727	0.747	0.996	0.103	0.633	0.663	0.992
		t^*_{OT}	0.093	0.584	0.580	0.911	0.231	0.655	0.635	0.961	0.144	0.664	0.711	0.991
10	100	t-bar	0.949	0.987	0.994	1.000	0.939	0.983	0.993	1.000	0.963	0.992	0.997	1.000
		F^*_{GT}	0.917	0.989	0.999	1.000	0.900	0.985	0.997	1.000	0.897	0.976	0.990	1.000
		F^*_{OT}	0.801	0.945	0.963	0.998	0.770	0.930	0.958	0.992	0.918	0.982	0.993	1.000
		K^*_{GT}	0.917	0.989	0.999	1.000	0.901	0.986	0.997	1.000	0.901	0.977	0.991	1.000
		K^*_{OT}	0.806	0.946	0.964	0.998	0.775	0.932	0.960	0.992	0.921	0.982	0.993	1.000
		t^*_{GT}	0.121	0.584	0.669	0.964	0.495	0.836	0.856	0.993	0.250	0.732	0.813	0.993
		t^*_{OT}	0.347	0.770	0.837	0.985	0.722	0.910	0.907	0.992	0.359	0.796	0.873	0.995

Table MS.M: Finite Sample Sizes for MA Errors, Demeaned Case

			(M1) θ_i ~U$(-0.8,-0.4)$				(M2) θ_i ~U$(-0.4,0.4)$				(M3) θ_i ~U$(0.4,0.8)$			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.290	0.450	0.434	0.638	0.087	0.107	0.103	0.152	0.091	0.100	0.100	0.115
		F^*_{GT}	0.117	0.179	0.174	0.275	0.046	0.057	0.057	0.071	0.046	0.056	0.053	0.067
		F^*_{OT}	0.090	0.142	0.139	0.219	0.047	0.064	0.065	0.084	0.045	0.057	0.057	0.068
		K^*_{GT}	0.116	0.180	0.175	0.276	0.046	0.057	0.057	0.069	0.045	0.056	0.053	0.068
		K^*_{OT}	0.090	0.142	0.139	0.217	0.048	0.064	0.065	0.084	0.045	0.057	0.057	0.069
		t^*_{GT}	0.125	0.172	0.161	0.258	0.052	0.060	0.059	0.074	0.046	0.058	0.058	0.068
		t^*_{OT}	0.105	0.139	0.137	0.200	0.051	0.063	0.061	0.074	0.049	0.059	0.059	0.073
10	100	t-bar	0.505	0.663	0.651	0.812	0.102	0.130	0.134	0.161	0.099	0.121	0.121	0.142
		F^*_{GT}	0.176	0.268	0.268	0.343	0.045	0.059	0.059	0.071	0.041	0.054	0.053	0.067
		F^*_{OT}	0.173	0.228	0.230	0.291	0.061	0.074	0.075	0.089	0.056	0.066	0.067	0.077
		K^*_{GT}	0.179	0.269	0.269	0.345	0.044	0.059	0.058	0.072	0.043	0.053	0.052	0.068
		K^*_{OT}	0.178	0.230	0.232	0.295	0.062	0.074	0.074	0.089	0.055	0.066	0.067	0.078
		t^*_{GT}	0.190	0.251	0.245	0.322	0.049	0.067	0.070	0.080	0.041	0.069	0.068	0.089
		t^*_{OT}	0.153	0.203	0.200	0.246	0.044	0.070	0.072	0.082	0.047	0.068	0.069	0.087

Table MP.M: Finite Sample Powers for MA Errors, Demeaned Case

			(M1) θ_i ~U$(-0.8,-0.4)$				(M2) θ_i ~U$(-0.4,0.4)$				(M3) θ_i ~U$(0.4,0.8)$			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.960	0.994	0.998	1.000	0.629	0.896	0.936	0.995	0.581	0.824	0.845	0.957
		F^*_{GT}	0.885	0.976	0.986	1.000	0.573	0.880	0.921	0.990	0.467	0.789	0.819	0.951
		F^*_{OT}	0.764	0.945	0.968	0.999	0.443	0.777	0.792	0.961	0.363	0.671	0.661	0.891
		K^*_{GT}	0.885	0.976	0.986	1.000	0.574	0.881	0.923	0.990	0.467	0.791	0.820	0.951
		K^*_{OT}	0.766	0.946	0.968	0.999	0.444	0.779	0.796	0.961	0.367	0.673	0.661	0.894
		t^*_{GT}	0.653	0.940	0.984	1.000	0.315	0.761	0.821	0.999	0.418	0.719	0.721	0.969
		t^*_{OT}	0.461	0.900	0.949	1.000	0.241	0.714	0.734	0.961	0.280	0.645	0.592	0.937
10	100	t-bar	1.000	1.000	1.000	1.000	0.964	0.993	0.998	1.000	0.851	0.961	0.972	0.998
		F^*_{GT}	0.997	1.000	1.000	1.000	0.931	0.989	0.996	1.000	0.833	0.948	0.969	0.995
		F^*_{OT}	0.991	0.999	1.000	1.000	0.844	0.966	0.984	0.998	0.727	0.890	0.913	0.981
		K^*_{GT}	0.997	1.000	1.000	1.000	0.932	0.989	0.996	1.000	0.835	0.949	0.970	0.995
		K^*_{OT}	0.991	0.999	1.000	1.000	0.847	0.967	0.984	0.998	0.734	0.893	0.917	0.982
		t^*_{GT}	0.878	0.984	0.998	1.000	0.474	0.860	0.951	0.994	0.560	0.828	0.856	0.994
		t^*_{OT}	0.942	0.991	0.999	1.000	0.735	0.929	0.946	0.996	0.571	0.871	0.875	0.989

Table TS.A: Finite Sample Sizes for AR Errors, Detrended Case

			(A1) ρ_i ~U$(-0.8, 0.8)$				(A2) ρ_i ~U$(0.2, 0.4)$				(A3) ρ_i ~U$(-0.8, 0.8)$, ID			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.076	0.086	0.084	0.102	0.068	0.087	0.085	0.108	0.054	0.066	0.068	0.076
		F^*_{GT}	0.034	0.049	0.049	0.060	0.037	0.050	0.050	0.060	0.039	0.050	0.050	0.060
		F^*_{OT}	0.041	0.053	0.052	0.075	0.043	0.053	0.052	0.070	0.042	0.054	0.056	0.063
		K^*_{GT}	0.034	0.049	0.049	0.060	0.037	0.050	0.050	0.060	0.039	0.050	0.050	0.060
		K^*_{OT}	0.041	0.053	0.052	0.075	0.043	0.053	0.052	0.070	0.042	0.055	0.056	0.063
		t^*_{GT}	0.033	0.045	0.045	0.059	0.039	0.052	0.051	0.062	0.039	0.047	0.048	0.058
		t^*_{OT}	0.034	0.048	0.049	0.059	0.038	0.052	0.052	0.067	0.032	0.050	0.050	0.064
10	100	t-bar	0.080	0.095	0.092	0.118	0.074	0.097	0.099	0.115	0.061	0.075	0.076	0.095
		F^*_{GT}	0.039	0.053	0.053	0.061	0.038	0.053	0.055	0.065	0.036	0.052	0.052	0.063
		F^*_{OT}	0.047	0.064	0.065	0.076	0.052	0.065	0.065	0.080	0.063	0.073	0.073	0.084
		K^*_{GT}	0.039	0.053	0.053	0.061	0.038	0.053	0.055	0.065	0.036	0.052	0.052	0.063
		K^*_{OT}	0.047	0.064	0.066	0.076	0.052	0.064	0.065	0.080	0.063	0.073	0.073	0.084
		t^*_{GT}	0.037	0.050	0.049	0.068	0.046	0.058	0.058	0.083	0.043	0.052	0.051	0.071
		t^*_{OT}	0.038	0.058	0.057	0.078	0.058	0.069	0.069	0.084	0.046	0.063	0.064	0.079

Table TP.A: Finite Sample Powers for AR Errors, Detrended Case

			(A1) ρ_i ~U$(-0.8, 0.8)$				(A2) ρ_i ~U$(0.2, 0.4)$				(A3) ρ_i ~U$(-0.8, 0.8)$, ID			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.335	0.651	0.655	0.911	0.324	0.633	0.670	0.875	0.313	0.663	0.672	0.928
		F^*_{GT}	0.240	0.626	0.677	0.899	0.225	0.599	0.633	0.857	0.224	0.580	0.609	0.866
		F^*_{OT}	0.176	0.485	0.471	0.777	0.169	0.453	0.445	0.723	0.247	0.612	0.637	0.894
		K^*_{GT}	0.240	0.626	0.677	0.899	0.225	0.599	0.633	0.857	0.224	0.580	0.609	0.866
		K^*_{OT}	0.176	0.485	0.471	0.777	0.169	0.453	0.445	0.723	0.247	0.612	0.638	0.894
		t^*_{GT}	0.038	0.331	0.282	0.874	0.199	0.487	0.459	0.852	0.059	0.391	0.342	0.867
		t^*_{OT}	0.052	0.342	0.338	0.672	0.146	0.419	0.410	0.759	0.080	0.430	0.392	0.879
10	100	t-bar	0.707	0.884	0.899	0.980	0.682	0.864	0.883	0.968	0.720	0.897	0.918	0.993
		F^*_{GT}	0.636	0.877	0.895	0.984	0.586	0.847	0.863	0.963	0.580	0.815	0.825	0.965
		F^*_{OT}	0.534	0.778	0.804	0.958	0.499	0.739	0.772	0.895	0.636	0.865	0.887	0.988
		K^*_{GT}	0.636	0.877	0.895	0.984	0.586	0.847	0.863	0.963	0.580	0.815	0.825	0.965
		K^*_{OT}	0.534	0.778	0.804	0.957	0.499	0.739	0.772	0.895	0.636	0.865	0.887	0.988
		t^*_{GT}	0.062	0.280	0.293	0.710	0.331	0.604	0.626	0.872	0.165	0.422	0.425	0.755
		t^*_{OT}	0.248	0.501	0.508	0.767	0.495	0.714	0.699	0.905	0.248	0.537	0.548	0.841

Table TS.M: Finite Sample Sizes for MA Errors, Detrended Case

			(M1) θ_i ∼U$(-0.8,-0.4)$				(M2) θ_i ∼U$(-0.4,0.4)$				(M3) θ_i ∼U$(0.4,0.8)$			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.599	0.755	0.732	0.906	0.135	0.183	0.182	0.235	0.164	0.179	0.178	0.208
		F^*_{GT}	0.120	0.159	0.159	0.192	0.051	0.071	0.072	0.089	0.046	0.063	0.064	0.078
		F^*_{OT}	0.085	0.112	0.110	0.143	0.058	0.080	0.082	0.099	0.047	0.069	0.067	0.084
		K^*_{GT}	0.120	0.159	0.159	0.192	0.051	0.071	0.072	0.089	0.046	0.063	0.064	0.078
		K^*_{OT}	0.085	0.112	0.110	0.143	0.058	0.080	0.082	0.099	0.047	0.069	0.067	0.084
		t^*_{GT}	0.176	0.225	0.218	0.315	0.061	0.071	0.071	0.085	0.057	0.070	0.070	0.087
		t^*_{OT}	0.124	0.162	0.161	0.219	0.054	0.074	0.070	0.097	0.052	0.067	0.068	0.083
10	100	t-bar	0.857	0.937	0.948	0.988	0.228	0.262	0.255	0.325	0.228	0.246	0.249	0.265
		F^*_{GT}	0.171	0.219	0.227	0.281	0.060	0.072	0.071	0.092	0.051	0.063	0.061	0.076
		F^*_{OT}	0.152	0.193	0.195	0.229	0.088	0.104	0.102	0.119	0.079	0.088	0.087	0.102
		K^*_{GT}	0.171	0.219	0.227	0.281	0.060	0.072	0.071	0.092	0.051	0.063	0.061	0.076
		K^*_{OT}	0.152	0.194	0.196	0.229	0.088	0.104	0.102	0.119	0.079	0.088	0.087	0.102
		t^*_{GT}	0.293	0.354	0.356	0.416	0.077	0.091	0.090	0.108	0.080	0.095	0.097	0.110
		t^*_{OT}	0.263	0.299	0.290	0.344	0.082	0.109	0.108	0.136	0.084	0.097	0.098	0.109

Table TP.M: Finite Sample Powers for MA Errors, Detrended Case

			(M1) θ_i ∼U$(-0.8,-0.4)$				(M2) θ_i ∼U$(-0.4,0.4)$				(M3) θ_i ∼U$(0.4,0.8)$			
N	T	tests	min	mean	med	max	min	mean	med	max	min	mean	med	max
5	100	t-bar	0.954	0.991	0.997	1.000	0.489	0.788	0.825	0.965	0.466	0.694	0.708	0.865
		F^*_{GT}	0.568	0.834	0.867	0.963	0.326	0.680	0.710	0.905	0.268	0.544	0.554	0.780
		F^*_{OT}	0.412	0.757	0.760	0.932	0.296	0.581	0.576	0.832	0.235	0.459	0.443	0.681
		K^*_{GT}	0.568	0.834	0.867	0.963	0.326	0.680	0.710	0.905	0.268	0.544	0.554	0.780
		K^*_{OT}	0.412	0.757	0.760	0.932	0.296	0.581	0.576	0.832	0.235	0.459	0.443	0.682
		t^*_{GT}	0.468	0.842	0.910	0.995	0.214	0.576	0.592	0.954	0.275	0.506	0.475	0.831
		t^*_{OT}	0.362	0.770	0.814	0.974	0.195	0.524	0.542	0.828	0.203	0.436	0.400	0.727
10	100	t-bar	0.999	1.000	1.000	1.000	0.840	0.956	0.972	0.996	0.796	0.911	0.927	0.972
		F^*_{GT}	0.891	0.963	0.976	0.999	0.674	0.902	0.940	0.982	0.527	0.780	0.821	0.911
		F^*_{OT}	0.874	0.969	0.982	0.998	0.640	0.869	0.903	0.978	0.527	0.749	0.768	0.887
		K^*_{GT}	0.891	0.963	0.976	0.999	0.674	0.902	0.940	0.982	0.527	0.780	0.821	0.911
		K^*_{OT}	0.874	0.969	0.982	0.998	0.640	0.869	0.903	0.978	0.528	0.749	0.768	0.887
		t^*_{GT}	0.783	0.942	0.984	0.996	0.370	0.691	0.778	0.920	0.464	0.679	0.663	0.916
		t^*_{OT}	0.914	0.967	0.983	0.999	0.612	0.806	0.827	0.964	0.600	0.750	0.748	0.907