Asset Demands of Heterogeneous Consumers with Uninsurable Idiosyncratic Risk

Peter Hartley
Rice University
and
The Australian National University

and

Chris Jones
The Australian National University

Abstract

We examine asset market equilibrium in an intertemporal economic model with individual and aggregate uncertainty and where the asset market is incomplete. Modigliani-Miller leverage irrelevance holds, even when consumers face borrowing constraints, because individual firms cannot alter the equilibrium portfolio of securities available to consumers. We show that households demand less risky portfolios as their financial wealth increases because a given asymmetry in asset holdings imparts more variability to income when wealth is high. Finally, we confirm previous results that endogenous rates of time preference, uninsurable idiosyncratic risk and household borrowing constraints produce a very low risk-free real interest rate.

Version: August 20, 1999

Asset Demands of Heterogeneous Consumers with Uninsurable Idiosyncratic Risk

Models used to analyze household saving and portfolio allocation, corporate financial policy and equilibrium asset prices within an intertemporal setting often assume that there is no production, no uncertainty or that optimizing agents are identical (for example, see Uzawa (1968), Becker (1980) and Lucas and Stokey (1982)). There is considerable doubt, however, whether representative agent models can account for the evidence on consumption and saving, asset prices and portfolio allocations (for example, see Deaton (1991), Cochrane and Hansen (1992) and Aiyagari (1994)).

Some authors have argued that uninsurable income risk and household borrowing constraints might enable the models to account for some of the evidence. Since these modifications make asset markets incomplete for investors, however, their effects have usually been examined in endowment economies. We examine the effects of uninsurable income risk and household borrowing constraints within a general equilibrium *production* economy. Even though the capital market is incomplete for investors, Modigliani-Miller leverage irrelevance holds in our model. Although household choices result in a unique *aggregate* debt-equity ratio, the financial policies of individual firms do not affect their market values because no single firm can alter the set of securities available to households.[1]

The uninsurable income shocks lead to differences in household wealth. Since we assume all households have the same constant relative risk averse utility function, we might expect households to choose portfolios with a level of risk that was independent of wealth. We find, however, that *households desire less risky portfolios as their financial wealth increases*. Household behavior depends on *total* wealth, which includes the (state independent) capitalized value of *expected* labor income. As financial wealth increases, it becomes a larger proportion of total wealth. Households with an indirect utility of total wealth function that has the same concavity as their utility of consumption function would make the proportionate changes in consumption and total wealth the same. They therefore would reduce the riskiness of their portfolios of financial assets as financial wealth increases.

We nevertheless find that even the wealthiest households choose end of period financial wealth that is *more variable across aggregate states than is consumption*. The capitalized value of expected labor income is constant across states. A given proportional variation in total wealth across states therefore requires more than a proportional variation in financial wealth.

We confirm the findings of Heaton and Lucas (1992) that households use financial assets to self-insure against income shocks. They smooth consumption by saving when there is a good shock to their income and dissaving when there is a bad shock to it.[2] In fact, households smooth consumption more against idiosyncratic risk than against aggregate uncertainty. Since idiosyncratic risk is fully

1. This is the basis for Fisher separation holding in our model. When firms face short-selling constraints, the capital market may be incomplete for both households and firms, and Fisher separation can fail. The objective function of the firm then depends on the intertemporal consumption preferences of its shareholders.

2. Household saving thus mitigates the welfare effects of the lack of formal insurance. Dixit (1987, 1989) argues that asymmetric information is likely to restrict the availability of private insurance against individual risk. The same problems do not arise for aggregate uncertainty because it can be observed.

diversifiable across households, changes in wealth to insure against idiosyncratic risk are offsetting in aggregate. In contrast, the attempt to save against aggregate income shocks affects net saving and therefore alters the equilibrium asset prices in a way that tends to discourage self-insurance.

Our results also have implications for the equity premium puzzle (Mehra and Prescott (1985)). We confirm evidence in previous studies that uninsured idiosyncratic income risk in the presence of credit or borrowing constraints (Aiyagari (1994), Aiyagari and Gertler (1991) and Hartley (1994)), and non-separable utility (Constantinides[3] (1990) and Ferson and Constantinides (1991)) reduce the risk-free interest rate. For plausible values of risk aversion and income uncertainty, however, they do not greatly increase the risk premium on equity.[4]

There are other reasons for relaxing the assumption of time separability in the utility function. Becker (1980) argues that when households have additive utility with different constant rates of time preference, in a long-run steady state all the capital will be owned by the household with the lowest rate of time preference. If more than one household shares this low rate of time preference then the distribution of capital across these households will be indeterminate. We confirm this result in our model.

Household borrowing constraints can eliminate this problem (Heaton and Lucas (1992), Aiyagari (1994) and Hartley (1994)). Another approach, however, is to allow time-interdependencies in household utility functions while retaining a constant rate of time preference (Ryder and Heal (1973), and Constantinides (1990)). Alternatively, the rate of time preference can be endogenized (Uzawa (1968), Epstein (1987), Epstein and Hynes (1991) and Shi and Epstein (1993)).[5]

We assume consumers become more impatient as wealth increases. This approach draws directly from Epstein and Hynes (1983), who consider a class of utility functions (in continuous time) where the rate of time preference depends positively on an index of future consumption. These functions are weakly additively separable because the marginal rates of substitution depend only on current and future, but not past, consumption. We use wealth as the index of future consumption because it greatly simplifies calculation of the household value function.

The indirect utility of wealth function becomes more concave when the rate of time preference is an increasing function of wealth. The increased concavity helps produce an equilibrium cross-sectional wealth distribution when households do not face borrowing constraints. The increased concavity also

3. Constantinides relaxes time separability by introducing *habit persistence*. Other modifications to time discounting have also been proposed as explanations for the asset pricing puzzles. For example, Benninga and Protopapadakis (1990) allow the household discount rate to exceed unity. Kocherlakota (1990) shows that, if consumption grows, positive interest rates may exist in infinite horizon growth economies where individuals have discount factors larger than one.

4. The results in Hartley (1994) suggest that including banks in the economy might yield different results.

5. Shi and Epstein (1993) compare the effects of habit formation when it enters the utility function directly versus when it makes the rate of time preference endogenous.

affects equilibrium asset prices. As we noted above, however, for the parameterization we examine, the quantitative magnitude of the effect is not large.

Finally, our model has implications for the literature on corporate financial policy. Auerbach and King (1983) and Dammon (1988) examine the financing decisions of firms in general equilibrium models where consumers face short-selling constraints and where there are corporate and personal taxes on security returns. They find that risk preferences move consumers away from the strict tax clienteles of the Miller (1977) certainty equilibrium. In their two period models, risk averse consumers hold more risky portfolios when they have higher wealth. High wealth consumers therefore prefer equity as a result of both tax and risk considerations. In our model, however, risk averse consumers hold *less* risky portfolios at higher wealth levels. Attitudes to risk in the two period models of Auerbach and King and Dammon depend on the concavity of the utility of consumption function. By contrast, in our multi-period model, they depend on the concavity of the indirect utility of wealth function. Our results suggest that the conflict between tax and risk preferences is likely to be more pronounced in a multi-period model with endogenous rates of time preference.

1. Model structure

The fundamentals in the economy are household preferences, the production technology, the 'financial technology' (the structure of the asset market used to transfer savings from households to firms and return the proceeds to investors) and the structure of shocks to the economy.

We assume there is a single undiversifiable aggregate shock to firm productivity that affects both labor income and the return to capital. The positive correlation between the aggregate component of labor income and the return to capital makes equities a risky investment for households.

Household incomes also are affected by serially independent[6] idiosyncratic shocks. The idiosyncratic shocks are uncorrelated across households, and the number of households is large enough that the aggregate of the idiosyncratic shocks equals the mean shock of zero in each period. To rule out market insurance of the idiosyncratic income shocks, we assume the value of these shocks cannot be verified. Since asset markets are incomplete, the equilibrium is not Pareto optimal. Thus, we have to solve for the equilibrium explicitly rather than solving an equivalent "planning problem."

1.1 The household utility function

As noted in the introduction, we relax the assumption of time separability of household preferences. Ryder and Heal (1973) introduced habit persistence by including an index of past consumption in

6. The results obtained by Aiyagari (1994) suggest that the effect of borrowing constraints on the equilibrium riskless interest rate could be enhanced by making the idiosyncratic income shocks serially correlated.

the utility function. Uzawa (1968) made the rate of time preference a function of current and future consumption, while Shi and Epstein (1993) include past, current and future consumption in the rate of time preference.[7] Epstein and Hynes (1991) examine recursive utility where the rate of time preference is an increasing function of future consumption. We adopt a similar approach by making the rate of time preference an increasing function of wealth.[8] Specifically, we assume household utility is

$$\sum_{t=1}^{\infty} \beta(W_{t-1})^t U(c_t) \tag{1}$$

where the time discount factor $\beta(W_{t-1}) \in (0,1)$ is a function of household wealth at the beginning of the period and $U(c)$ is the utility of current consumption. Since all households face the same distribution of labor income, differences in financial wealth capture differences in expected future consumption opportunities. Making the discount factor a function of wealth rather than some other index of future (or past) consumption simplifies the numerical analysis by allowing us to use one state variable to describe the consumer's maximization problem.

1.2 Production

Let K be the capital stock, and L_0 and L_1 the employment by the representative firm in states $\varepsilon = \varepsilon_0$ and $\varepsilon = \varepsilon_1$ respectively where the values taken by ε in each period:

$$\varepsilon = \begin{cases} \varepsilon_0 \text{ with probability } \pi \\ \varepsilon_1 \text{ with probability } 1-\pi \end{cases} \tag{2}$$

are common to all firms and where $\varepsilon_1 > \varepsilon_0$ and the mean of ε is equal to 1.0. Use w_0 and w_1 to denote the real wages in states $\varepsilon = \varepsilon_0$ and $\varepsilon = \varepsilon_1$. Assume the cash flows of the firm in each state are:

$$S_0 = \varepsilon_0 K^{\alpha} L_0^{1-\alpha} + (1-\delta)K - w_0 L_0 \tag{3}$$

$$S_1 = \varepsilon_1 K^{\alpha} L_1^{1-\alpha} + (1-\delta)K - w_1 L_1 \tag{4}$$

The first order conditions for employment in each state yield

$$(1-\alpha)\varepsilon_0 K^{\alpha} L_0^{-\alpha} = w_0 \tag{5}$$

$$(1-\alpha)\varepsilon_1 K^{\alpha} L_1^{-\alpha} = w_1 \tag{6}$$

---

7. Shi and Epstein show how the dynamics implied by each of these specifications differ and, in particular, how their approach can result in cyclical wealth accumulation.
8. The appendix discusses a recursive utility formulation due to Epstein and Zin (1989) and Epstein (1992). While this specification is theoretically superior, it proved to be much harder to work with numerically.

In equilibrium, *per capita* labor demand has to equal the fixed *per capita* labor supply of 1 so

$$y_0 = w_0 = (1-\alpha)\varepsilon_0 k^{\alpha} \tag{7}$$

$$y_1 = w_1 = (1-\alpha)\varepsilon_1 k^{\alpha} \tag{8}$$

where k is the *per capita* capital stock of the representative firm and y_0 and y_1 are per capita labor incomes in the aggregate states 0 and 1. The revenues of the representative firm in the two aggregate states, net of labor costs, can be expressed in per capita terms as:

$$x_0 = \varepsilon_0 k^{\alpha} + (1-\delta)k - y_0 \tag{9}$$

$$x_1 = \varepsilon_1 k^{\alpha} + (1-\delta)k - y_1 \tag{10}$$

1.3 Asset markets

Security market traders, who face nothing more than a zero net wealth constraint, can issue derivative securities based on the assets issued by firms. Since firms experience two states, the aggregate state space of asset payoffs available to security market traders can be spanned using any two linearly independent securities. If households are unconstrained in their asset trades over the aggregate state space, they also can span the aggregate state space with two linearly independent securities.

We also examine constraints that prevent households from shorting equities or holding assets with negative returns in any state.[9] Since such constraints prohibit households from making future contributions to firms, they can be thought of as solvency constraints. When households are constrained, their asset holdings cannot span the aggregate state space. We show that firms and security market traders can minimize the impact of the constraints by supplying a full set of primitive securities.

1.3.1 Spanning of the State-Space by Firms and Security Traders

The formal analysis allows for S aggregate states. The examples apply the results where $S = 2$.

Assumption 1: There are no barriers to entry into the capital market. Security traders have "equal access" in the sense that each of them can issue any derivative security based on the assets issued by firms. Therefore, all derivative securities must be priced independently of the trader.

Lemma 1: The capital market can offer any pattern of returns if the N securities issued by firms, with returns defined over the S states, have an $(S \times N)$ state-contingent payout matrix R with rank = S.

Proof: By constructing a portfolio τ, traders can create derivative securities D where:

9. More generally, we could allow households to borrow no more than some fraction of their minimum labor income. We may also want to restrict households to borrowing at an interest rate that exceeds the rate paid by firms.

$$R\tau \equiv \underset{S\times N}{\begin{bmatrix} R_{11} & \dots & R_{1N} \\ \vdots & \ddots & \vdots \\ R_{S1} & \dots & R_{SN} \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_N \end{bmatrix}} = \underset{S\times 1}{\begin{bmatrix} D_1 \\ \vdots \\ D_S \end{bmatrix}} \equiv D \tag{11}$$

When rank(R) = S, and each τ_n can be positive or negative, security traders can create a full set of primitive securities, and therefore a security with any pattern of returns over the S states.

Example: Suppose there are 2 states, s = 0 and 1, and two securities, κ_1 and κ_2, with payout matrix:

$$R \equiv \underset{S\times N}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \tag{12}$$

(so that rank(R) = N = S = 2). To create the primitive security D_0, which pays only in state 0, $\tau_1 = 1$ and $\tau_2 = -1$. The primitive security D_1, which pays only in state 1 already exists as security 2, so $\tau_1 = 0$ and $\tau_2 = 1$. Thus, traders can create a full set of primitive securities.

By assumption 1, every security issued by firms or traders has a perfect substitute. Firms and security traders therefore are price-takers in the capital market. This leads directly to:

Lemma 2: (*No-Arbitrage Condition*) If the state-contingent payout matrix for firm securities has rank(R) = S, then the riskless bond price p_B and the primitive securities prices p_s satisfy:

$$\sum_{s=1}^{S} p_s = p_B \tag{13}$$

Proof: This follows from assumption 1 and lemma 1. A unit of the riskless bond is a perfect substitute for a portfolio combining one unit of each primitive security. Since there are no-arbitrage profits in a competitive equilibrium, the riskless bond must sell at the sum of the prices of the primitive securities.

Comment: For the 2 state example, the price p_B of a riskless bond with identical returns in each state, and the prices p_0 and p_1 of the pure state-contingent claims (D_0 and D_1 above) satisfy $p_0+p_1 = p_B$.

1.3.2 Spanning of the State-Space by Consumers

When consumers have equal access and are unconstrained in their security trades, and when the state-contingent payout matrix for firm securities has rank(R) = S, consumers can span the aggregate state space.[10] Short-selling constraints, however, limit the set of derivative securities D consumers can hold.

They cannot hold negative quantities of any security issued by firms or offered by traders. Furthermore, each security available to them must have non-negative returns in every state. This leads to:

Lemma 3: Consumers facing short-selling constraints cannot span the state space of asset returns even though the state-contingent payout matrix for firm securities has rank(R) = S.

Proof: Suppose there are M derivative securities $\{D(m) = R\,\tau(m),\ m = 1,\ldots, M\}$ that are available to consumers and let $D = [D_{sm}] = [D(1)\ D(2)\ \ldots\ D(M)]$ denote the $S \times M$ matrix of state-contingent payouts on these derivative securities. As a result of the short-selling constraints, the elements of D must satisfy $D_{sm} \geq 0$. The set of portfolios available to consumers then consists of all assets Q that satisfy:

$$D\sigma = Q$$

subject to $$\sigma_m \geq 0 \text{ for all } m$$

and $$D_{sm} \geq 0 \text{ for all } s \in S \text{ and all } m.$$

The returns on all portfolios Q must therefore form a proper subset of the state space. Thus, consumers cannot span the state space even if the set of securities issued by firms has rank S.

Two State Example

To see how short-selling constraints restrict consumers in the 2 state case, consider the payout matrix:

$$D\sigma \equiv \begin{bmatrix} d_0 & 1 \\ 1 & d_1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = Q \tag{14}$$

with $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $1 \geq d_0 \geq 0$ and $1 \geq d_1 \geq 0$. The values of the state-contingent returns, d_0 and d_1, determine the size of the state space consumers can span. If $d_0 d_1 < 1$, at least one derivative security available to households is risky. The matrix D has full rank and the vector of asset demands is equal to:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \frac{1}{d_0 d_1 - 1} \begin{bmatrix} d_1 Q_0 - Q_1 \\ d_0 Q_1 - Q_0 \end{bmatrix}. \tag{15}$$

The requirement $\sigma \geq 0$ implies $d_0 \leq Q_0/Q_1 \leq 1/d_1$. Consider the following three cases for different values of d_0 and d_1, that is, for different degrees of riskiness in the derivative securities.

(1) Riskless Debt and One Risky Security

The returns to the riskless bond are measured along the 45° line κ_B, while the returns to the risky securities are measured along the lines κ_2 and κ_1 in the left and right hand panels of Figure 1.

10.The proof of this is identical to the proof of Lemma 1 for firms.

The shaded regions of the diagrams are the returns to household portfolios composed of positive amounts of the two derivative assets. The shaded region in the left panel of Figure 1 measures the attainable returns when security 1 is riskless (with $d_0 = 1$) and security 2 is risky (with $0 \le d_1 < 1$). The shaded region in the right panel measures the attainable returns when security 1 is risky (with $0 \le d_0 < 1$) and security 2 is riskless (with $d_1 = 1$). Both these regions satisfy the short-selling constraints.

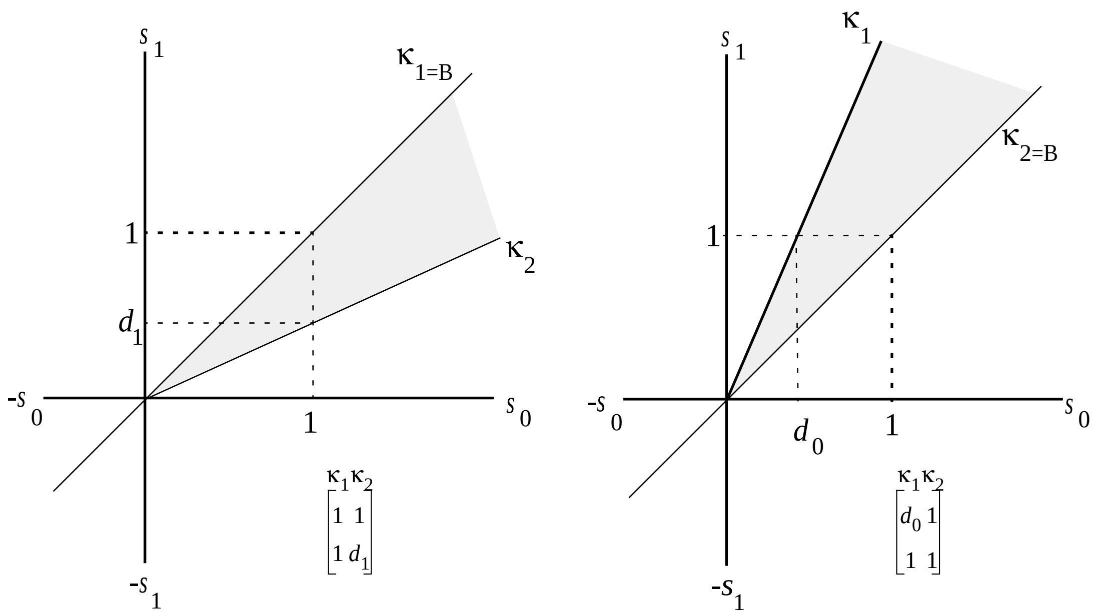

FIGURE 1. Attainable Returns with Riskless Debt

(2) Two Risky Securities with Positive Returns in Both States

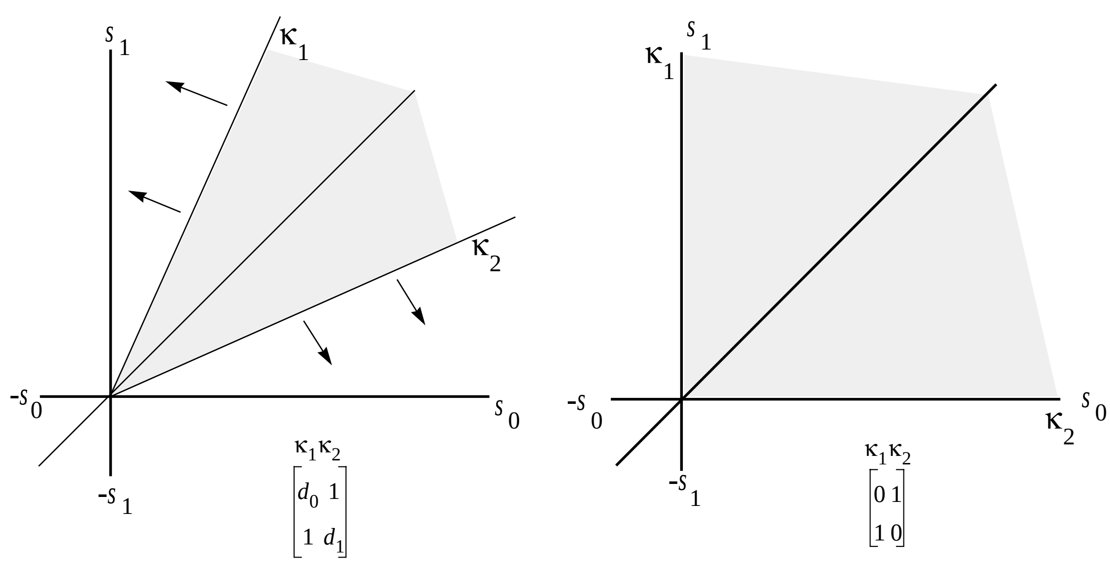

FIGURE 2. Attainable Returns with Two Risky Securities

The left panel of Figure 2 is the combination of the two shaded regions in Figure 1. It illustrates the attainable returns when there is no riskless bond and both derivative securities have positive returns in each state ($0 \le d_0 < 1$ and $0 \le d_1 < 1$). As d_0 and d_1 approach zero, the set of attainable returns increases as the line segments rotate toward the s_0 and s_1 axes, respectively.

(3) Two Primitive Securities

In the limit $d_0 = d_1 = 0$, the two derivative securities are primitive securities and the attainable returns are the entire positive state space, illustrated in the right panel of Figure 2. The following lemma justifies a focus on this case later in the paper.

Lemma 4: In the two-state case (14) with $1 \ge d_0 \ge 0$ and $1 \ge d_1 \ge 0$, the short-selling constraints $\sigma_1 \ge 0$, $\sigma_2 \ge 0$, affect consumers as little as possible when $d_0 = d_1 = 0$.

Proof: Consumers with access to the returns in the right panel of Figure 2 could limit their choices to the left panel of Figure 2. Expanding their choice set cannot make them worse off. Any set of returns that strictly contains the set in the right panel of Figure 2 will violate the short-selling constraints.

Comment. In a competitive capital market with no transactions costs, security traders have an incentive to provide assets that minimize the effect of the constraints. Note that when consumers hold both securities in amounts σ_1 and σ_2 they effectively hold $\min(\sigma_1, \sigma_2)$ of riskless debt.

2. Firm maximization

When firms and security traders have equal access to a competitive capital market there is always a perfect substitute for every security they offer. This leads to:

Theorem 1: (*Modigliani-Miller Leverage Irrelevance*) For the structure of production in section 1, if the capital market is competitive the value of the firm is independent of the level of debt financing.

Proof. The firm has cash flows of x_0 in state 0 and x_1 in state 1. Since $\varepsilon_1 > \varepsilon_0$, $x_1 > x_0$ and the firm can pay x_0 in *both* states and an additional $x_1 - x_0$ in state 1. Alternatively, the firm (or security traders) could "unbundle" the x_0 units of *debt* into λx_0 units of debt and $(1-\lambda)x_0$ units of each state claim.[11] The combined *equity* security would have returns $(1-\lambda)x_0$ in state 0, $x_1 - \lambda x_0$ in state 1 and value:

$$\lambda x_0 p_B + (1-\lambda)x_0 p_0 + (x_1 - \lambda x_0)p_1 = x_0 p_B - (1-\lambda)x_0 p_B + (1-\lambda)x_0 p_0 + (x_1 - x_0)p_1 + (1-\lambda)x_0 p_1 \quad (16)$$

But the no-arbitrage condition in lemma 2 implies $(1-\lambda)x_0 p_0 + (1-\lambda)x_0 p_1 = (1-\lambda)x_0 p_B$. Hence the market value of the cash flows is independent of λ:

$$x_0 p_B + (x_1 - x_0)p_1 = x_0 p_0 + x_1 p_1 \quad (17)$$

11. Note that, since firms or security traders can short assets, we do not require $0 \le \lambda \le 1$.

Comment. This result holds whether or not consumers face short-selling constraints. Consumers will demand a unique aggregate asset portfolio to satisfy risk and intertemporal consumption preferences. If one firm changes its financial structure, other firms will be induced by incipient movements in asset prices to take the offsetting position.[12] Aivazian and Callen (1987) observe this adjustment process in an endowment economy with a Miller (1977) tax structure. We conjecture that the result will also hold in our production economy augmented by their tax structure.

Financial policy irrelevance, and the absence of any leverage related costs in a common information setting, implies that firms will maximize profits (that is, *Fisher Separation* holds).

Theorem 2: For the structure of production in section 1, the optimal capital stock is given by

$$k^{\alpha-1} = \frac{1-p_B(1-\delta)}{\alpha(p_0\varepsilon_0+p_1\varepsilon_1)} \equiv \frac{\delta+r}{\alpha(p_0\varepsilon_0+p_1\varepsilon_1)(1+r)} \tag{18}$$

where r is the *riskless* rate of interest.
Proof. The *net* value of the firm is

$$x_0p_0+x_1p_1-k = p_0(\varepsilon_0k^{\alpha}+(1-\delta)k-y_0)+p_1(\varepsilon_1k^{\alpha}+(1-\delta)k-y_1)-k \tag{19}$$

The firm chooses k to maximize (19), leading to a first order condition

$$(p_0\varepsilon_0+p_1\varepsilon_1)\alpha k^{\alpha-1}+(1-\delta)(p_0+p_1)-1 = 0 \tag{20}$$

But from (13)

$$p_0+p_1 = p_B = \frac{1}{1+r} \tag{21}$$

Substituting (21) into (20) and rearranging we obtain (18).

When firms are price takers in the capital market the right hand side of (18) is unaffected by the value of the capital stock chosen by any one firm.

Comment. If there is no uncertainty with $\varepsilon_0 = \varepsilon_1 = \bar{\varepsilon}$ then (18) will reduce to

$$\bar{\varepsilon}\alpha k^{\alpha-1} = \delta+r \tag{22}$$

The left side of (22) is the marginal product of capital, while the right side is the risk-free user cost of capital. Both sides of the equation are independent of the debt-capital ratio.

When $\varepsilon_1 > \varepsilon_0$, firms must pay financiers a risk premium. The premium is embodied in the state-con-

12.In the absence of transactions costs, offsetting changes by other firms (or security traders) take place instantly.

tingent prices that make the right side of (18) greater than $r + \delta$. If we now let $\bar{\varepsilon}$ denote the mean of the productivity shock ε, we can define an *implicit risk premium* Φ by

$$p_0\varepsilon_0 + p_1\varepsilon_1 \equiv \frac{\bar{\varepsilon}}{(1+r)}\left(\frac{\delta + r}{\delta + r + \Phi}\right)$$

that is

$$\Phi = (\delta + r)\left[\frac{\bar{\varepsilon}}{(1+r)(p_0\varepsilon_0 + p_1\varepsilon_1)} - 1\right] \tag{23}$$

and (18) becomes

$$\bar{\varepsilon}\alpha k^{\alpha - 1} = \delta + r + \Phi \tag{24}$$

3. Household budget constraint and household maximization

Household labor income is given by:

$$y = \begin{cases} y_0 + z_0 \text{ with probability } \pi\theta \\ y_0 + z_1 \text{ with probability } \pi(1-\theta) \\ y_1 + z_0 \text{ with probability } (1-\pi)\theta \\ y_1 + z_1 \text{ with probability } (1-\pi)(1-\theta) \end{cases} \tag{25}$$

where $y_0 = \varepsilon_0(1-\alpha)k^\alpha$ and $y_1 = \varepsilon_1(1-\alpha)k^\alpha$ are the *aggregate* components of labor income (common across households). Each household also draws an idiosyncratic shock to labor income z with $z_1 > z_0$. The number of households is large enough that the *sample* mean of z each period equals the *population* mean of the distribution, $\theta z_0 + (1-\theta)z_1$, which, for convenience, is taken to be zero. The distribution of z is independent of the distribution of ε. The value of ε and the values of z for each household are revealed after households have chosen assets for t. Households then receive interest, dividend payments and labor income, and choose their consumption for period t. Wealth available at the end of the period is allocated to financial assets in preparation for next period's consumption. Consumption and end of period wealth in the state $\varepsilon = \varepsilon_i$ and $z = z_j$, $i, j = 0, 1$ are denoted by c_{ij} and W_{ij}.

Applying lemma 4 we can, without loss of generality, assume households can hold only pure state contingent claims. Use p_0 and p_1 for the prices of these state contingent securities. Let $\kappa_0(W)$ and $\kappa_1(W)$ be the number of claims to consumption in states 0 and 1 purchased by a representative household with wealth W. When choosing assets, a household with wealth W faces a budget constraint

$$p_0\kappa_0(W) + p_1\kappa_1(W) = W. \tag{26}$$

Define the *fraction* $\rho(W)$ of returns in period t+1 that accrue in state 0 by

$$\rho(W) = \frac{\kappa_0(W)}{\kappa_0(W)+\kappa_1(W)} \tag{27}$$

so that $1-\rho(W)$ the *fraction* that accrue in state 1. We then have:

Lemma 5: For a portfolio allocation $\rho(W_{t-1})$ of financial wealth W_{t-1}, financial wealth in period t is

$$W_t = \begin{cases} \kappa_0(W_{t-1}) = \dfrac{\rho(W_{t-1})W_{t-1}}{\rho(W_{t-1})p_0+(1-\rho(W_{t-1}))p_1} \text{ in state 0} \\ \kappa_1(W_{t-1}) = \dfrac{(1-\rho(W_{t-1}))W_{t-1}}{\rho(W_{t-1})p_0+(1-\rho(W_{t-1}))p_1} \text{ in state 1} \end{cases} \tag{28}$$

Proof: Solve equations (26) and (27) for $\kappa_0(W)$ and $\kappa_1(W)$.

Comment. The denominator in (28) represents the current price of a security that pays off ρ units of consumption in state 0 and $(1-\rho)$ units of consumption in state 1. The number of such securities owned by a household investing wealth W_t will be $W_t/(\rho p_0+(1-\rho)p_1)$.

Lemma 6: The short-selling constraints can be written as

$$0 \le \rho(W) \le 1 \text{ for all } W > 0, \tag{29}$$

$$W_{00} = y_0 + z_0 + \frac{\rho W}{\rho p_0 + (1-\rho)p_1} - c_{00} \ge 0 \tag{30}$$

$$W_{01} = y_0 + z_1 + \frac{\rho W}{\rho p_0 + (1-\rho)p_1} - c_{01} \ge 0 \tag{31}$$

$$W_{10} = y_1 + z_0 + \frac{(1-\rho) W}{\rho p_0 + (1-\rho)p_1} - c_{10} \ge 0 \tag{32}$$

$$W_{11} = y_1 + z_1 + \frac{(1-\rho) W}{\rho p_0 + (1-\rho)p_1} - c_{11} \ge 0 \tag{33}$$

Proof: The short-selling constraints restrict the asset and goods market trades of households to ensure that households are holding non-negative financial wealth at all times and in all states of the world. In particular, financial wealth $W \ge 0$ in all periods, while the budget constraint (26) together with (21) imply $\kappa_0(W)/W$ and $\kappa_1(W)/W$ must lie in the unit interval. Furthermore, if $\kappa_0(W) = 0$ then $\kappa_1(W) > 0$ and conversely.

4. Risk Neutral Households

We show that when households are risk neutral the state-contingent asset prices p_0 and p_1 are constant and can be treated simply as parameters. When households are risk averse, asset prices become *functions* of the current aggregate state of the economy.

Definition: When p_0 and p_1 are constant, the household *value function* $V(W)$ is the solution to the functional equation:[13]

$$V(W) = \max_{\rho} \beta(W)\left\{\pi\theta \max_{c_{00}, W_{00}} [U(c_{00}) + V(W_{00})] + \pi(1-\theta) \max_{c_{01}, W_{01}} [U(c_{01}) + V(W_{01})]\right. \tag{34}$$

$$\left. + (1-\pi)\theta \max_{c_{10}, W_{10}} [U(c_{10}) + V(W_{10})] + (1-\pi)(1-\theta) \max_{c_{11}, W_{11}} [U(c_{11}) + V(W_{11})]\right\}$$

where the maximizations are constrained by (29)–(33). We shall use φ_{ij} for the Lagrange multiplier on the constraint in state $\varepsilon = \varepsilon_i$ and $z = z_j$, $i, j = 0, 1$ in (30)–(33), and μ_0 and μ_1 for the multipliers for the constraints $\rho \geq 0$ and $\rho \leq 1$.

Theorem 3: Household portfolio allocation over the two pure state contingent claims, $\rho(W)$, and maximizing consumptions and end of period assets satisfy (29)–(33) together with:

$$U'(c_{00}) = V'(W_{00}) + \varphi_{00} \text{ with } \varphi_{00}\left(y_0 + z_0 + \frac{\rho W}{\rho p_0 + (1-\rho)p_1} - c_{00}\right) = 0 \tag{35}$$

$$U'(c_{01}) = V'(W_{01}) + \varphi_{01} \text{ with } \varphi_{01}\left(y_0 + z_1 + \frac{\rho W}{\rho p_0 + (1-\rho)p_1} - c_{01}\right) = 0 \tag{36}$$

$$U'(c_{10}) = V'(W_{10}) + \varphi_{10} \text{ with } \varphi_{10}\left(y_1 + z_0 + \frac{(1-\rho)W}{\rho p_0 + (1-\rho)p_1} - c_{10}\right) = 0 \tag{37}$$

$$U'(c_{11}) = V'(W_{11}) + \varphi_{11} \text{ with } \varphi_{11}\left(y_1 + z_1 + \frac{(1-\rho)W}{\rho p_0 + (1-\rho)p_1} - c_{11}\right) = 0 \tag{38}$$

$$\frac{(\beta(W)p_0 p_1 W)}{(\rho p_0 + (1-\rho)p_1)^2}\left\{\frac{\pi[\theta U'(c_{00}) + (1-\theta)U'(c_{01})]}{p_0} - \frac{(1-\pi)[\theta U'(c_{10}) + (1-\theta)U'(c_{11})]}{p_1}\right\} = \mu_1 - \mu_0 \tag{39}$$

$$\mu_0\rho = 0, \mu_1(1-\rho) = 0, \mu_0 \geq 0, \rho \geq 0, \mu_1 \geq 0, 1 \geq \rho \tag{40}$$

Proof: For a given ρ, the first order conditions for the maximizing choices of consumptions and end

13. We assume sufficient conditions are placed on $U(c)$ and $\beta(W)$ to guarantee there is a unique solution to (34). In the discussion below we calculate solutions for $V(W)$ for specific functional forms of $U(c)$ and $\beta(W)$.

of period wealth levels subject to the constraints (30)–(33) are given by (30)–(33) and (35)–(38). Now let M_{00}, M_{01}, M_{10} and M_{11} denote the solutions to these maximization problems in each of the states. Then the maximization problem for the choice of ρ can be written:

$$\max_{\rho} \beta(W)[\pi\theta M_{00} + \pi(1-\theta)M_{01} + (1-\pi)\theta M_{10} + (1-\pi)(1-\theta)M_{11}] + \mu_0\rho + \mu_1(1-\rho) \tag{41}$$

Now observe that if $\varphi_{ij} = 0$ then $U'(c_{ij}) = V'(W_{ij})$ and

$$\frac{dM_{0j}}{d\rho} = \frac{p_1 U'(c_{0j})W}{(\rho p_0 + (1-\rho)p_1)^2} \tag{42}$$

$$\frac{dM_{1j}}{d\rho} = -\frac{p_0 U'(c_{1j})W}{(\rho p_0 + (1-\rho)p_1)^2} \tag{43}$$

while if $\varphi_{ij} > 0$, c_{ij} is determined by the budget constraints (30)–(33) and $M_{ij} = U(c_{ij}) + V(0)$ so that again the derivatives of M_{ij} are given by (42)–(43). Hence, the first order condition for the choice of ρ subject to the constraints (29) is given by (39) and (40).

Theorem 4: If $U(c) = c$, and there are no short-selling constraints, equilibrium asset prices are

$$p_0 = \frac{\pi}{1+r} \text{ and } p_1 = \frac{1-\pi}{1+r} \tag{44}$$

while the maximizing consumptions and end of period wealth levels satisfy

$$1 = U'(c_{ij}) = V'(W_{ij}). \tag{45}$$

Proof: If there are no short-selling constraints, ρ is not constrained to lie in the unit interval and, from (40), $\mu_0 = \mu_1 = 0$. Equation (39) for the optimal value of ρ then implies

$$\frac{\pi}{p_0} = \frac{1-\pi}{p_1} \tag{46}$$

But from (21),

$$p_0 + p_1 = \frac{1}{1+r} \tag{47}$$

where r is the riskless interest rate. From (46) and (47), the prices of a claim to a unit of consumption in states 0 and 1 are given by (44). In the absence of short-selling constraints, (30)–(33) also are irrelevant so $\varphi_{ij} = 0$ and the first order conditions for the optimal consumption (35)–(38) become (45).

Comment. By re-scaling consumption, we can ensure $U' = 1$ for any risk neutral household.

Corollary 1: If $U(c) = c$, there are no short-selling constraints, and $V'(W)$ is monotonic, then

$W_{ij} = W^*$ for all i and j, c_{01} - c_{00} = c_{11} - c_{10} = z_1 - z_0, so the difference in consumption when only idiosyncratic income varies matches the difference in income, and expected consumption equals:

$$\pi(\theta c_{00} + (1-\theta)c_{01}) + (1-\pi)(\theta c_{10} + (1-\theta)c_{11}) = W(1+r) - W^* + \{\pi y_0 + (1-\pi)y_1 + \theta z_0 + (1-\theta)z_1\} \quad (48)$$

Proof: When $V'(W)$ is monotonic, (45) implies W_{ij} is constant across states. The remaining results follow from (44) and the budget constraints (30)–(33).

Corollary 2: If $U(c) = c$, and there are no short-selling constraints the equilibrium k satisfies (22).

Proof: Substituting (44) into (18) we obtain:

$$k^{\alpha-1} = \frac{\delta + r}{\alpha(\pi\varepsilon_0 + (1-\pi)\varepsilon_1)} \quad (49)$$

which can be re-arranged into (22) where $\bar{\varepsilon}$ is interpreted as the mean of ε.

Theorem 5: If $U(c) = c$, there are no short-selling constraints, and β is independent of W then the equilibrium riskless real rate of interest is given by

$$\beta(1+r) = 1 \quad (50)$$

and the household value function is given by

$$V(W) = W + \frac{\beta}{1-\beta}\{\pi y_0 + (1-\pi)y_1 + \theta z_0 + (1-\theta)z_1\} \quad (51)$$

that is, financial wealth plus the discounted expected value of labor income.

Proof: Applying the envelope theorem to the functional equation (34),

$$V'(W) = \frac{\beta}{\rho p_0 + (1-\rho)p_1}\{\rho[\pi\theta V'(W_{00}) + \pi(1-\theta)V'(W_{01})] + (1-\rho)[(1-\pi)\theta V'(W_{10}) + (1-\pi)(1-\theta)V'(W_{11})]\} \quad (52)$$

Substituting (45) into (52) we obtain

$$V'(W) = \frac{\beta}{\rho p_0 + (1-\rho)p_1}\{\rho\pi + (1-\rho)(1-\pi)\} \quad (53)$$

Substituting the state contingent asset prices (44) into (53) and using (45) we conclude that

$$1 = V'(W) = \beta(1+r) \quad (54)$$

It is easy to verify that when asset prices satisfy (44) and (50), $V(W)$ given by (51) solves (34).

Comment. Theorem 5 implies that, when households are risk neutral and have an identical constant rate of time preference, equilibrium asset prices are determined by household behavior. The riskless real rate of interest r equals the household rate of time preference and the price of consumption in state i is the probability of state i discounted at the rate r. The supply of savings is perfectly elastic at these rates of return and *average* household wealth and the per capita capital stock are determined by the demand for loans from firms, (22). While aggregate household wealth is given by the demand for loans from firms, the distribution of that wealth across households is indeterminate. Households are indifferent between consuming in different periods or states when asset prices satisfy (44) and (50).

As noted in the introduction, we can avoid this indeterminacy result by assuming households have a non-constant rate of time preference:

Theorem 6: If $U(c) = c$, there are no short-selling constraints, and β is a differentiable function of W with a functional form that ensures $V(W)$ is increasing in W but strictly concave, then $W_{ij} = W^*$ for all i and j and the household value function is given by

$$V(W) = [A + W(1+r)]\beta(W) \tag{55}$$

where A and W^* jointly solve

$$A = \frac{\pi y_0 + (1-\pi)y_1 + \theta z_0 + (1-\theta)z_1 - W^*[1-\beta(W^*)(1+r)]}{1-\beta(W^*)} \tag{56}$$

$$\beta(W^*)(1+r) + [A + W^*(1+r)]\beta'(W^*) = 1\,. \tag{57}$$

Proof: Apply the envelope theorem to functional equation (34) when β depends on W to get

$$\begin{aligned} V'(W) = {} & \frac{\beta(W)}{\rho p_0 + (1-\rho)p_1}\{\rho[\pi\theta V'(W_{00}) + \pi(1-\theta)V'(W_{01})] + \\ & (1-\rho)[(1-\pi)\theta V'(W_{10}) + (1-\pi)(1-\theta)V'(W_{11})]\} + \frac{\beta'(W)}{\beta(W)}V(W) \end{aligned} \tag{58}$$

Now use (44) and (45) from Theorem 4 to conclude

$$V'(W) = \beta(W)(1+r) + \frac{\beta'(W)}{\beta(W)}V(W) \tag{59}$$

Equation (59) has a solution for $V(W)$ of the form (55) for a constant A. When $V(W)$ is concave, V' is monotonic and the corollary to Theorem 4 implies $W_{ij} = W^*$ for all i and j and the expected utility of current consumption is given by (48). Substituting (48) and (55) into (34) we conclude that the indirect utility of wealth $V(W)$ will be given by (55) if A and W^* jointly solve (56) and (57).

Comment. Since asset prices satisfy (44) and households are risk neutral, the distribution of state claims across households is indeterminate. Now, however, the distribution of wealth is determinate.

Households adjust consumption to compensate for differences in their initial wealth W and choose the same final wealth W^*. The riskless real interest rate r and household wealth W^* are jointly determined in equilibrium so per capita demand for loans from firms, (22), equals per capita supply of loans from households W^*. Once k has been determined, the supply of *state contingent* claims also will be determined and this will have to equal the *aggregate* per capita demand from households.

Numerical solution for a particular $\beta(W)$

It will be useful to compare the solution to this model to the solutions of models with risk averse households in the next section of the paper. We now assume that $\beta(W)$ has the form

$$\beta(W) = \frac{1}{1+\xi+\psi W} \tag{60}$$

Then $V'(W) > 0$ for $A\psi < (1+r)(1+\xi)$, in which case $V''(W) < 0$. For the parameter values in Table 1,

TABLE 1. Parameter values for the risk neutral model

Parameter	ψ	ξ	π	θ	z_0	z_1	ε_0	ε_1	α	δ
Value	0.0025	0.05/0.95	0.4	0.2	-0.1	0.025	0.85	1.1	0.25	0.03

the approximate solution is A = 18.530052, $W^* = k$ = 2.213988 and r = 0.107739. The approximate equilibrium incomes in the two aggregate states are y_0 = 0.777632 and y_1 = 1.006347. The indirect utility of wealth function $V(W)$, together with its first and second derivatives, is given for $1.5 \le W \le 3$ in Table 2. It is interesting to note that the equilibrium riskless real interest rate is considerably *larger* than $(1-\beta)/\beta$ for typical values of W. This follows from (57) and the fact that $\beta'(W) < 0$.

TABLE 2. Equilibrium discount factor, value function and derivatives

W	$\beta(W)$	$W)$	$V'(W)$	$V''(W)$
1.50	0.946628	19.113985	1.003382	-0.004749
1.75	0.946068	19.364682	1.002196	-0.004746
2.00	0.945509	19.615083	1.001012	-0.004744
2.25	0.944950	19.865188	0.999830	-0.004741
2.50	0.944393	20.114998	0.998650	-0.004738
2.75	0.943836	20.364513	0.997472	-0.004735
3.00	0.943279	20.613734	0.996296	-0.004732

5. Risk averse households

Now suppose households have a constant relative risk averse utility of consumption function

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}. \tag{61}$$

If the time discount factor β, income and asset prices were all constant, when there are no short-selling constraints the functional equation (34) would have a solution of the form (for A constant)

$$V(W) = A\left[\frac{y}{r} + W\right]^{1-\gamma} \tag{62}$$

When β is a function of W, and income varies, $V(W)$ must be approximated numerically. The following solutions are for $\gamma = 1.75$, the remaining parameter values in Table 1 and $\beta(W)$ given by (60).

Unlike the risk neutral case, where consumption immediately adjusts to allow wealth to achieve the target W^*, households with decreasing marginal utility of consumption adjust their wealth gradually. The budget constraints (30)–(33) now produce four stochastic difference equations. The per capita capital stock held by firms also adjusts according to a stochastic difference equation. The result will be a stationary *distribution* of k not a stationary *value* for the capital stock.

Since the ε shock affects all households simultaneously, the cross-sectional wealth distribution, Ω, will vary with the sequence of realizations for ε. In effect, Ω becomes another multi-dimensional “state variable.” Household indirect utility becomes a function of Ω. Furthermore, when solving their maximization problems, rational households would forecast the evolution of Ω as a function of the possible realizations for future values of ε. It is, however, impossible to make Ω a “state variable” for the maximization problem. It may also be quite unreasonable to assume households know Ω or how it evolves in response to aggregate shocks.

We consider a simpler “bounded rationality” model. Specifically, we assume that, when forming expectations, households characterize the aggregate state of the economy by the *per capita* capital stock (or per capita household wealth) and use linear approximations to the state transition equations for k in the two possible aggregate states, ε_0 and ε_1:

$$k_{t0} = A_{k0} + B_{k0}k_{t-1} \text{ and } k_{t1} = A_{k1} + B_{k1}k_{t-1} \tag{63}$$

For this reason, and also because the aggregate components of labor income, given by (7) and (8), will now also be functions of k_{t-1}, the household value function will also become a function of the current per capita capital stock k_{t-1}. In place of (34), the household value function will now satisfy:

$$V(W_{t-1}, k_{t-1}) = \max_{\rho}\beta(W_{t-1})\Big\{\pi\theta \max_{c_{00}, W_{00}}[U(c_{00}) + V(W_{00}, A_{k0} + B_{k0}k_{t-1})] + \tag{64}$$

$$\pi(1-\theta)\max_{c_{01}, W_{01}}[U(c_{01}) + V(W_{01}, A_{k0} + B_{k0}k_{t-1})] + (1-\pi)\theta\max_{c_{10}, W_{10}}[U(c_{10}) + V(W_{10}, A_{k1} + B_{k1}k_{t-1})] +$$

$$(1-\pi)(1-\theta)\max_{c_{11}, W_{11}}[U(c_{11}) + V(W_{11}, A_{k1} + B_{k1}k_{t-1})]\Big\}$$

The choice variables c_{ij}, W_{ij} and ρ satisfying the first order conditions (35)–(40) will also be functions of $k_{t\text{-}1}$ and $W_{t\text{-}1}$. Capital market equilibrium will require:

$$x_0 = \alpha\varepsilon_0 k_{t-1}^{\alpha} + (1-\delta)k_{t-1} = \int \kappa_0(W_{t-1}, k_{t-1}) d\Omega(W_{t-1}, k_{t-1}) \tag{65}$$

$$x_1 = \alpha\varepsilon_1 k_{t-1}^{\alpha} + (1-\delta)k_{t-1} = \int \kappa_1(W_{t-1}, k_{t-1}) d\Omega(W_{t-1}, k_{t-1}) \tag{66}$$

where $\kappa_0(W_{t-1},k_{t-1})$ and $\kappa_1(W_{t-1},k_{t-1})$ are given in (28) and where $\Omega(W_{t-1},k_{t-1})$ is the current *actual* cross-sectional distribution of wealth. The representative firm's choice of capital at the end of period t–1 will also satisfy (18):

$$k_{t-1}^{\alpha-1} = \frac{1-(p_{0t-1}+p_{1t-1})(1-\delta)}{\alpha(p_{0t-1}\varepsilon_0 + p_{1t-1}\varepsilon_1)} \tag{67}$$

One could, in principle, solve this system for *any* constants in (63) and any initial k_{t-1}. Households could only be said to be "boundedly rational", however, when the functions in (63) approximate the non-linear relationships between k_{t-1} and k_t around a "representative" value of k. We use a two-step numerical procedure to find an approximate solution to the model when $k_{t-1} = k^*$, the mean of the stationary distribution of per capita capital.

First, we solve the model numerically assuming k is constant, and is expected to remain constant, at some value $\bar{k}$. At $\bar{k}$, we require the corresponding cross-sectional wealth distribution Ω to satisfy, for any subset of wealth levels A:

$$Pr(W \in A) = \int_A d\Omega = \pi\theta \int_{W_{00}(W)\in A} d\Omega + \pi(1-\theta) \int_{W_{01}(W)\in A} d\Omega + (1-\pi)\theta \int_{W_{10}(W)\in A} d\Omega + (1-\pi)(1-\theta) \int_{W_{11}(W)\in A} d\Omega \tag{68}$$

and

$$\bar{k} = \int W d\Omega \tag{69}$$

while the asset prices p_0 and p_1 also ensure that (65) and (66) are valid for $\bar{k}$ and Ω.

In the second set of iterations, we linearly approximate the difference equations for the evolution of mean wealth to obtain starting expressions for (63). Using these linear approximations, and the solutions for the (constant) asset prices in the "static expectations" model, we solve for a two-dimensional value function (64) defined on a grid of W and k values. In the process, we obtain corresponding functions for $W_{ij}(W, k)$. Using these functions, we perform a Monte Carlo simulation to arrive at an equilibrium distribution of k, with mean k^*. Finally, we examine the value function and the choice variables as a function of W evaluated at $k = k^*$.

5.1 No short-selling constraints

For small values of ψ, we expect $V(W)$ to have a functional form close to (62). Hence, we approximated $\log(-V(W))$ by a spline function in $\log(B + W)$ for a constant B that was determined iteratively. The algorithm we use is related to the algorithms discussed in more detail in Hartley (1994, 1995) and Judd (1992). In brief, it involves the steps:

1. Choose a grid of values for W at which $V(W)$ will be evaluated. In the results reported below, we used 160 values for W spread evenly between -20 and 20.

2. Choose an initial approximation $V_0(W)$ for $V(W)$.[14]

3. Guess the equilibrium asset prices p_0 and p_1.[15]

4. Solve the firm's first order condition (18) for k using the asset prices p_0 and p_1.

5. Calculate the aggregate component of per capita labor income y_0 and y_1 using (7) and (8).

6. Using the functional approximation for $V'(W)$, and noting that the multipliers φ_{ij}, μ_1 and μ_2 are all zero when there are no short-selling constraints, solve the first order conditions (35)–(38) and (39) for $c_{00}(W)$, $c_{01}(W)$, $c_{10}(W)$, $c_{11}(W)$ and $\rho(W)$ on the grid of values for W.[16]

7. Substitute the maximizing $c_{00}(W)$, $c_{01}(W)$, $c_{10}(W)$, $c_{11}(W)$ and $\rho(W)$ into the right hand side of (34) to obtain a new set of values $V_1(W)$ for $V(W)$. Fit a spline approximation

$$\sum_{j=0}^{6} a(n,j)[\log(W+B)-\zeta_n]^{6-j},\ n = 0, 1, \ldots, 9 \tag{70}$$

to the values for $\log(-V_1(W))$ for a constant B that was initially set at $\bar{y}/r$. There were nine interior breakpoints $\zeta_1,\ldots,\zeta_9$ in the spline approximation, with

$$\log(30+\zeta_{i+1})-\log(30+\zeta_i) = \frac{[\log(50)-\log(10)]}{10} \tag{71}$$

for $i = 0,\ldots,9$ and $\zeta_0 = -20$ and $\zeta_{10} = 20$. The approximation was constrained so that the second derivative was continuous across the interior breakpoints.[17]

8. Use the spline approximation to evaluate the *inverse* of the coefficient of absolute risk aversion $C_A^{-1}(W) = -V'(W)/V''(W)$. Regress $C_A^{-1}(W)$ against 1 and W to obtain a new estimate of B, and then re-calculate the spline approximation. Use the new approximation to $V(W)$ to re-evaluate the right hand side of (34) for the same maximizing $c_{00}(W)$, $c_{01}(W)$, $c_{10}(W)$, $c_{11}(W)$ and $\rho(W)$. Fit a

---

14.We used (62) with income equal to its expected value and r equal to the household discount rate when W=0.

15.We started with p_0 and p_1 given by (44).

16.The equations are solved using the MatLab Optimization Toolbox routine *fsolve*.

17.We used the least squares spline approximation routine *spap2* in the MatLab Spline Toolbox. Other routines in the MatLab Spline Toolbox were used to differentiate and evaluate the spline approximation to $V(W)$.

new spline approximation $V_2(W)$ to the resulting values for $V(W)$.
9. If the change in $V(W)$, $||V_2(W) - V_0(W)|| > 10^{-6}$, return to step 6. Otherwise, go to step 10.
10. Substitute the maximizing $c_{ij}(W)$ and $\rho(W)$ into the budget constraints (30)–(33) to obtain a stochastic difference equation for the evolution of wealth. The difference equation yields a Markov process on the intervals of wealth in the partition of W. Calculate the stationary cross-sectional wealth distribution for this Markov process by iterating the mapping until $||\Omega_{n+1}(W) - \Omega_n(W)|| < 10^{-6}$.
11. Substitute the maximizing $\rho(W)$ into (28), weight by the stationary cross-sectional wealth distribution $\Omega(W)$ and sum, to find the per capita demand for asset income, $\bar{\kappa}_0$ and $\bar{\kappa}_1$. The per capita supplies are given by (9) and (10). Adjust p_0 and p_1 in proportion to the excess demands

$$\Delta p_i = f(\bar{\kappa}_i - x_i) \tag{72}$$

unless the adjustment in both prices is less than 10^{-6}. The fraction f is chosen to stabilize the price adjustments. If the price adjustment exceeds 10^{-6}, return step 4 and use the new prices.
12. Fit linear approximations (about the final value for $\bar{k}$) to the final solutions $W_{ij}(W)$ for the evolution of household wealth.
13. Use these linear approximations on the right side of (64) together with the asset prices[18] $p_0(k)$ and $p_1(k)$, to determine a two-dimensional approximation to the function $V(W,k)$ for a grid of values for k. We also obtain new mappings $W_{ij}(W,k)$, $c_{ij}(W,k)$ and $\rho(W,k)$.
14. Use $W_{ij}(W,k)$ in a Monte Carlo simulation over 10000 periods to obtain a distribution for k and associated average cross-sectional wealth distributions $\Omega(W,k)$ at each value of k. Obtain the per capita demands for asset income, $\bar{\kappa}_0(k)$ and $\bar{\kappa}_1(k)$ for each k in the grid. Obtain new asset prices at each k by adjusting the previous values in proportion to the excess demands

$$\Delta p_i(k) = f[\bar{\kappa}_i(k) - x_i(k)] \tag{73}$$

Also, we use the pairs of values (k_{t-1}, k_t) from the last iteration of the Monte Carlo to obtain new regression estimates of the linear relationship between k_{t-1} and k_t. Return to step 13 unless the absolute change in asset prices and regression coefficients is less than 10^{-4}. The function estimates presented and discussed below are the final approximations evaluated at the mean value of k on the final iteration.

The final household value function

The final approximate value for B was 66.429631. The maximum absolute difference between the final value for $V(W)$ and the spline approximation at the grid of values for W is approximately 2.06×10^{-13} while the average absolute error at the 160 values of W is approximately 5.87×10^{-14}.

18.Starting with p_0 and p_1 indpendent of k and given by the solutions to the “static expectations” problem.

The values for $V(W)$, the spline approximations to $V'(W)$, $V''(W)$ and the spline approximation to the inverse of the coefficient of absolute risk aversion, $C_A^{-1}(W)$, are graphed in Figure 3. The final graph in Figure 3 also plots the difference between $C_A^{-1}(W)$ and the linear approximation to $C_A^{-1}(W)$ as a measure of the departure of $V(W)$ from (62). The units for $C_A^{-1}(W)$ are given on the left hand scale, while the units for the residuals from the linear regression are on the right hand scale.

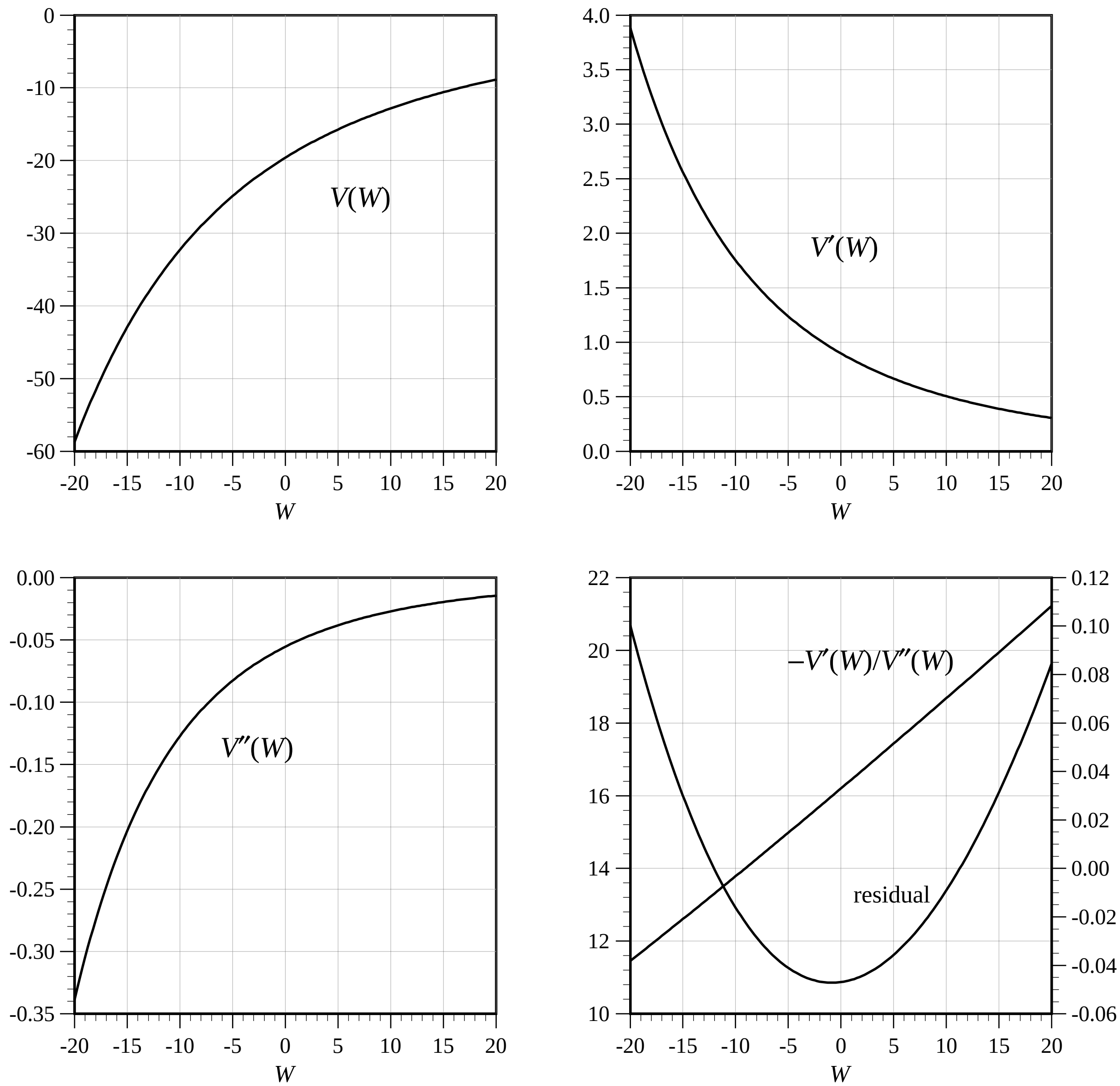

FIGURE 3. Value function in the unconstrained economy

Result 1. The value function is quite close to constant relative risk averse in $B + W$. The final graph in Figure 3 shows that $-V'(W)/V''(W)$ is approximately linear in W, with very small percentage departures from linearity at high and low values of W.

Result 2. The value function is more concave than the original utility function. The inverse, Γ, of the coefficient of W in the regression of $C_A^{-1}(W)$ against 1 and W is an estimate of the relative risk aversion in $V(W)$. Since the $\gamma = 1.75$ whereas $\Gamma = 4.088998$, we conclude that endogenous time discounting makes individuals behave as if they are considerably more risk averse than $U(c)$ would indicate.

Portfolio allocation

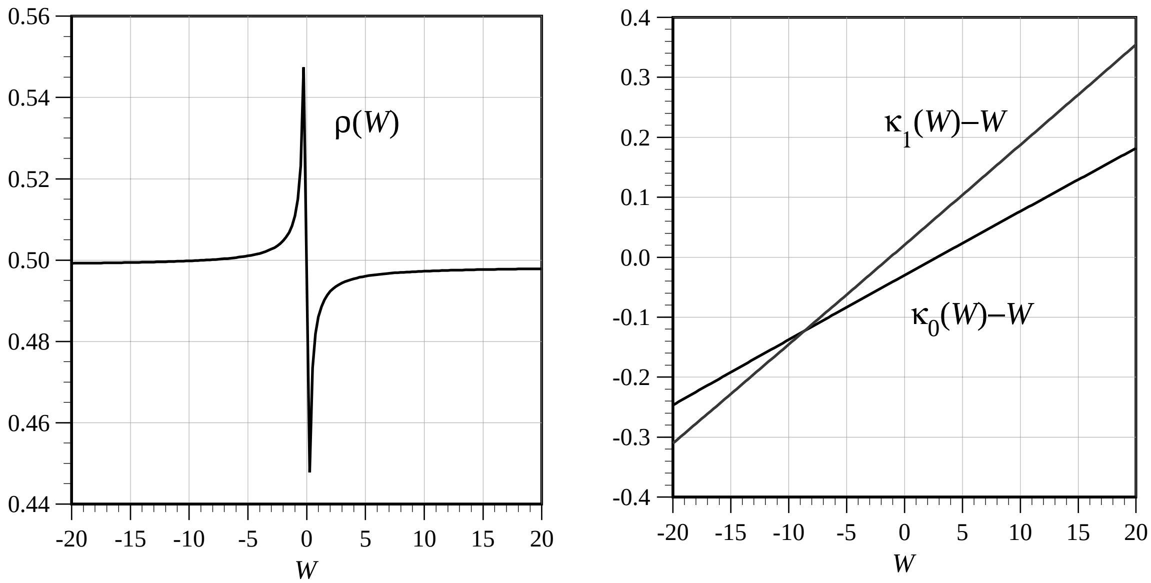

FIGURE 4. Portfolio allocation in the unconstrained economy

The proportion of wealth $\rho(W)$ allocated to state 0 consumption claims is graphed in the left panel of Figure 4. The allocation of wealth is irrelevant, and thus $\rho(W)$ is undetermined, when $W = 0$.[19] The function graphed in the left panel of Figure 4 is therefore discontinuous at $W = 0$.

While $\rho(W)$ is discontinuous at $W = 0$, the value of the portfolio in period t+1 in either state, $\kappa_0(W)$ or $\kappa_1(W)$, has the same limit as $W \to 0$ from above or below. This is illustrated in the right panel of Figure 4, which graphs the change in the value of the portfolio in aggregate states 0 and 1, $\kappa_0(W)-W$ and $\kappa_1(W)-W$. When $\rho(W)$ takes relatively large absolute values for W close to 0, W is small in absolute value so $\kappa_0(W)$ and $\kappa_1(W)$ have the same limit as W is approached from above or below.

Result 3. Households hold less risky asset portfolios as financial wealth increases above zero (or decreases from zero to about -12). A portfolio where $\rho(W) = 0.5$ is riskless since it consists entirely of bonds with a payout that is independent of the state. As ρ tends to either 0 or 1, portfolios consist entirely of assets that pay off in either of the aggregate income states 0 or 1.

19.For this reason, we omitted $W = 0$ from the grid of W values we used to approximate the solution.

If $V(W)$had been given by (62), it would have had the same concavity as $U(c)$. In that case, the first order conditions for household maximization (35)–(38) would have implied that proportional variations in *total* wealth, $y/r+W$, across states would have matched the proportional variations in consumption across states. The proportional variations in financial wealth alone, however, will exceed the proportional variations in $y/r+W$. Therefore, as W increases, the proportional variations in W would have to decrease to keep the proportional variations in $y/r+W$ and consumption the same. Hence, consumers would choose less risky portfolios of financial assets as W increases.

We noted that the indirect utility function $V(W)$ is in fact approximately constant relative risk averse in $B+W$, but more concave than the utility of consumption function. Since $V(W)$ is more concave than $U(c)$, the first order conditions (35)–(38) imply that the proportional variations across states in end-of-period $B+W$ will be *smaller* than the proportional variations in consumption. Nevertheless, for B sufficiently large, proportional variations in *financial* wealth will have to exceed proportional variations in consumption. The extent of this additional variability in W will decline as W becomes a larger proportion of $B+W$. Hence, we still obtain the result that portfolios become less risky as W increases.

Result 4. Asset income reinforces the asymmetry in labor income across aggregate states for almost all households. The right hand panel of Figure 4 illustrates the difference in asset *income* across states. It shows that when $W < 0$ and $\rho(W) > 0.5$, the household is *paying out* more in the low aggregate income state 0 than in the high aggregate income state 1. On the other hand, when $W > 0$, since $\rho(W) < 0.5$ the household is *receiving* more asset income in state 1 than in state 0. Asset income (or payments) therefore offset the asymmetry in labor income only for households deeply in debt.

The disparity in asset income across states depends not only on the *composition* of the portfolio but also on the total amount invested in financial assets. Specifically, expressions (28) for $\kappa_0(W)$ and $\kappa_1(W)$ imply that a given departure of $\rho(W)$ from 0.5 produces a larger disparity of asset income across states the higher the value of W. Thus, even though portfolios become *less risky* as W increases above zero, the disparity in asset income across states 0 and 1 *increases*.

In equilibrium, we must have $\bar{\kappa}_0 = x_0 < x_1 = \bar{\kappa}_1$ so that, *in the aggregate*, households must hold more state 1 claims than state 0 claims. The results in Figure 4 show that almost all households, however, have $\kappa_0 < \kappa_1$. As we shall see below, the disparities across aggregate states in both consumption and end of period financial wealth increase as W increases. In order to achieve this end, the disparity across aggregate states in current period asset income has to reinforce the disparity in labor income.

Consumption

The final maximizing consumption choices are illustrated in Figure 5. The consumption functions in the different states are very similar, so we have only graphed $c_{00}(W)$ along with various *differences* in consumption across states as a function of financial wealth *W*.

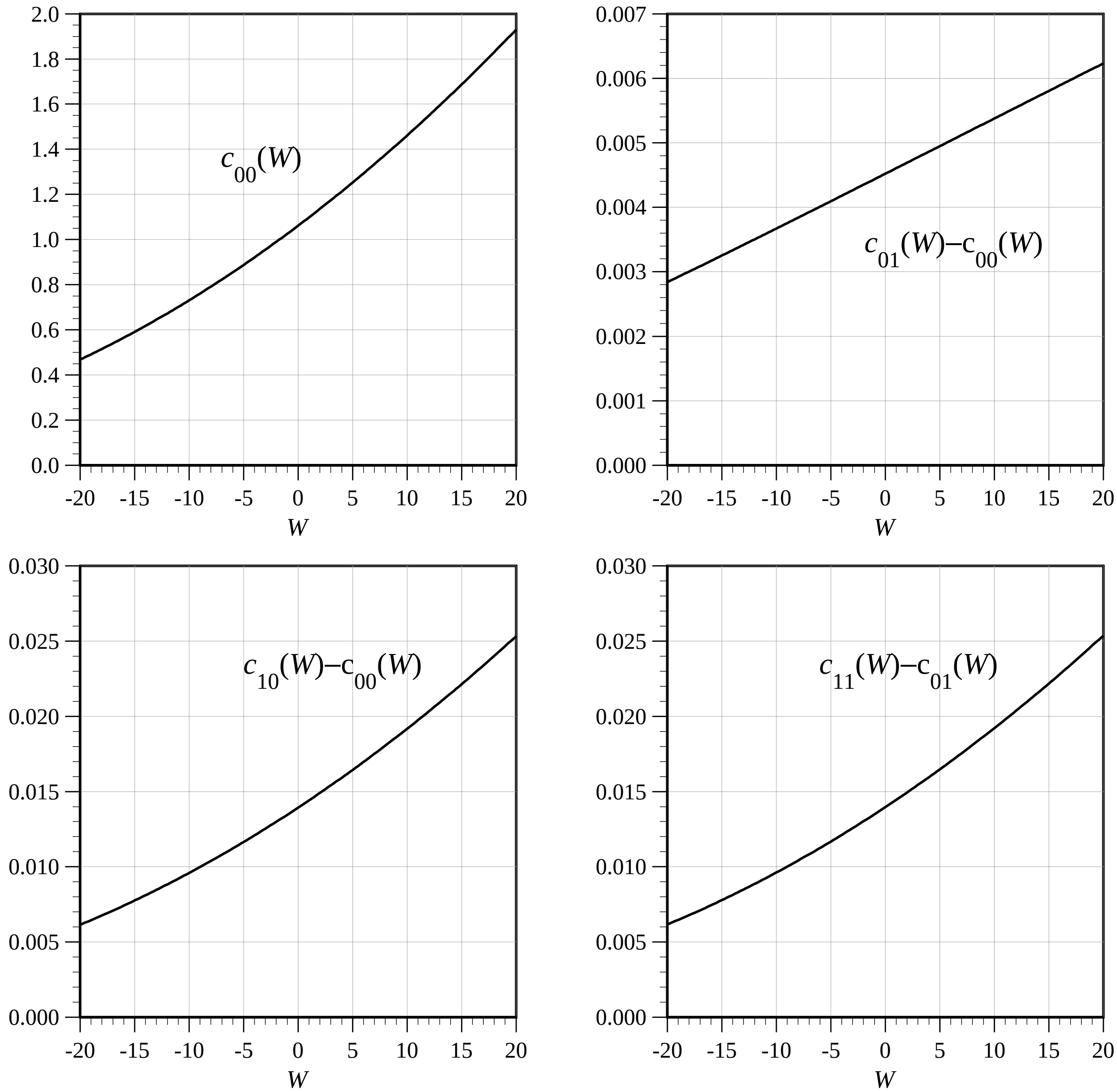

FIGURE 5. Consumption in the unconstrained economy

Result 5. Consumption, and the marginal propensity to consume out of W, increase with W in all states. As wealth increases, consumers have a higher rate of time discount and therefore increase their current consumption more than proportionately with the increase in wealth.

Result 6. The difference in consumption across idiosyncratic states is much less than the difference in income across the same states. The second graph in Figure 5 plots the difference in consumption across the two idiosyncratic states as a function of *W*, and when the aggregate state is zero.[20] The difference in income across these states is $z_1–z_0 = 0.125$. Since the consumption difference is much less than the income difference, households manage to "insure" against the idiosyncratic income shocks even though they cannot trade claims contingent on the occurrence of these states. They effectively "self-insure" by liquidating wealth in the low income state and saving in the high income state.

Result 7. Households also "insure" against aggregate income shocks, but to a lesser extent than they insure against idiosyncratic income shocks. The final two graphs in Figure 5 plot the differences of consumption across aggregate states as a function of W. The first graph applies to the case where the idiosyncratic shock is 0 (low income) in both cases, while the second graph is for the case where the idiosyncratic shock is 1 (high income) in both cases. The difference in income between the two aggregate states is approximately $y_1 - y_0 = 0.333851$. Since consumption differences across aggregate states fall short of the income differences, households also insure against aggregate income fluctuations.

While the difference in income across aggregate states is about 2.67 times the difference in income across idiosyncratic states, however, the consumption difference across aggregate states is at least 3 times the consumption difference across idiosyncratic states. Someone has to bear aggregate income risk. The attempt of households to redistribute income from state 1 to state 0 will be frustrated in equilibrium by a rise in p_0 relative to p_1.

In contrast, when households suffer a negative idiosyncratic shock and dissave, other households are enjoying a positive idiosyncratic shock and save some of the additional income. The desire to transfer income across idiosyncratic income states is not frustrated by movements in asset prices. Nevertheless, if assets contingent on the occurrence of idiosyncratic income shocks could be traded the outcome would be different. Idiosyncratic income shocks would then be *completely* diversifiable. Households could obtain full insurance without paying an interest premium and idiosyncratic income shocks would not affect consumption.

Result 8. As wealth increases, households undertake less self-insurance in the sense that the disparity in consumption across idiosyncratic states increases as financial wealth increases. This is illustrated in the second graph in Figure 5. The budget constraints in (30) and (31) imply $(W_{01}+c_{01})-(W_{00}+c_{00}) = z_1-z_0 = 0.125$. The change in $W_{01}-W_{00}$ as W increases therefore has to be equal in magnitude and opposite in sign to the change in $c_{01}-c_{00}$ as W increases:

20. The graph of the difference in consumption across the two idiosyncratic states when the aggregate state is one is very similar to the second graph in Figure 5 and has been omitted. Also note that the difference between the lines in the final two graphs in Figure 5 can be written as $c_{11}(W)-c_{10}(W)-[c_{01}(W)-c_{00}(W)]$.

$$\frac{d}{dW}(W_{01}-W_{00}) = -\frac{d}{dW}(c_{01}-c_{00}) \ . \tag{74}$$

In the absence of the inequality constraints in (30)–(33), the first order conditions (35)–(38) imply that $U'(c) = V'(W)$. Then since $U(c)$ is constant relative risk averse, and $V(W)$ is close to constant relative risk averse in $B+W$, the *proportional* variation in $B+W$ across states should approximate γ/Γ times the *proportional* variation in c, that is,

$$W_{01}-W_{00} \cong \frac{\gamma}{\Gamma}\left[\frac{W_{00}+B}{c_{00}}\right](c_{01}-c_{00}) = \frac{\gamma}{\Gamma}\left[\frac{1+B/W_{00}}{c_{00}/W_{00}}\right](c_{01}-c_{00}) \tag{75}$$

Since the ratio in square brackets on the right side of (75) declines as W increases, we can conclude, using (74), that $W_{01}-W_{00}$ declines, and $c_{01}-c_{00}$ increases, as W increases.

Result 9. The disparity in consumption across aggregate states also increases as W increases, and at a rate that exceeds the increase in disparity in consumption across idiosyncratic states. The final two graphs in Figure 5 show that $c_{10}-c_{00}$ and $c_{11}-c_{01}$ increase with W. Further, the rate of increase in these differences is higher than the rate of increase in $c_{01}-c_{00}$. The first order conditions for household maximization again imply that the proportional variation in $B+W$ across states should approximate γ/Γ times the proportional variation in c. The budget constraints now imply, however, that the difference across aggregate states in the sum of consumption and end of period wealth will equal $y_1-y_0+\kappa_1(W)-\kappa_0(W)$, which increases with W. Hence, $(W_{10}-W_{00})+(c_{10}-c_{00})$ and $(W_{11}-W_{01})+(c_{11}-c_{01})$ must now both increase with W, so $c_{10}-c_{00}$ and $c_{11}-c_{01}$ increase faster than $c_{01}-c_{00}$ and $c_{11}-c_{10}$. Also, the first graph in Figure 6 shows that, while the disparities across *idiosyncratic* states in end of period wealth, $W_{01}-W_{00}$ and $W_{11}-W_{10}$, decrease with increases in W, the disparities across *aggregate* states, $W_{10}-W_{00}$ and $W_{11}-W_{01}$, increase with W.

The wealth distribution

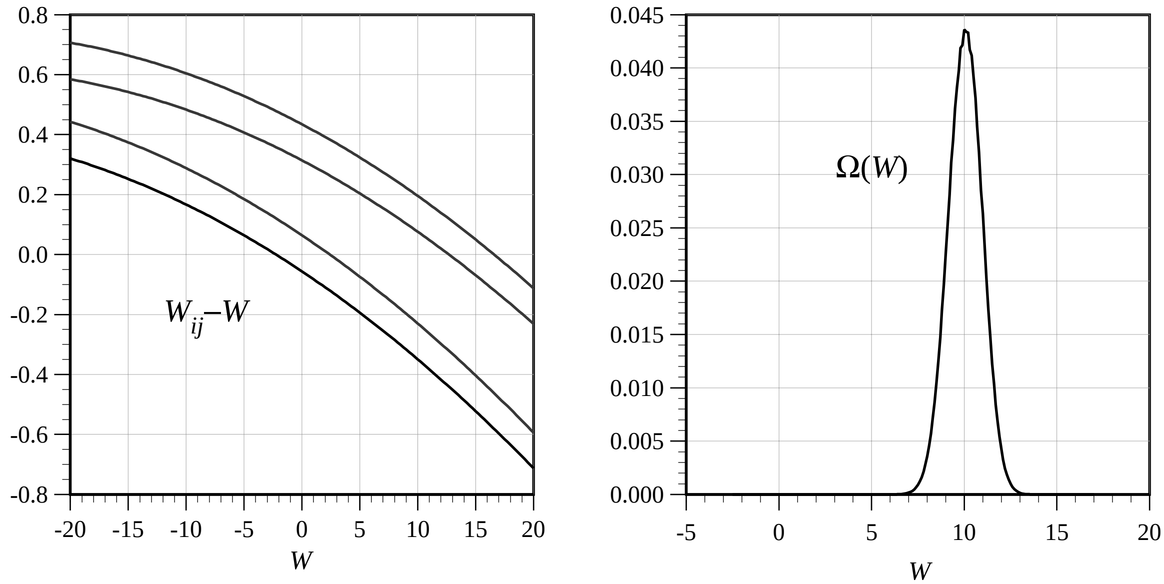

FIGURE 6. Stochastic difference equation for wealth in the unconstrained economy

The maximizing choices for $\rho(W)$ and $c_{ij}(W)$ lead, upon substitution into (30)–(33), to a set of four difference equations for the evolution of household wealth. The graphs of $W_{ij}(W)$ against W are virtually indistinguishable from a 45˚ line since $W_{ij}(W) - W$ is very small relative to W at most wealth levels. The left panel of Figure 6 therefore plots $W_{ij}(W) - W$ against W. The end of period wealth increases from the low income states (i=0, j=0 and i=0, j=1) to the high income states (i=1, j=0 and i=1, j=1).

It is clear from the graph in the left panel of Figure 6 that the mappings $W \to W_{ij}(W)$ each have single fixed point W_{ij}^*. *If asset prices were to remain constant, and were expected to remain constant*, then household wealth would evolve according to these difference equations. Regardless of the initial cross-sectional distribution, the final cross-sectional distribution of wealth would eventually end up in the interval $[W_{00}^*, W_{11}^*]$. The right panel in Figure 6 graphs the stationary cross-sectional wealth distribution corresponding to the stochastic difference equation represented in the left panel of Figure 6.

Asset prices

The cross-sectional wealth distribution in the right panel of Figure 6 was used with the solution for the maximizing $\rho(W)$ to determine the per capita demand for asset income in states 0 and 1, $\bar{\kappa}_0$ and $\bar{\kappa}_1$. The asset prices p_0 and p_1 were chosen to ensure this demand matched the state-contingent per capita supply of asset income from firms.

Result 10. The risk-free interest rate is well below the rate of time preference for any of the households holding wealth in the interval $[W_{00}^*, W_{11}^*]$. Approximate solutions for the asset prices were $p_0 = 0.399793$ and $p_1 = 0.586180$. From (21), the implied risk-free interest rate $r \cong 0.014227$ as opposed to

0.107739 in the risk neutral model. When households become risk averse they have a dramatically increased incentive to save as a form of "self-insurance." This increased saving leads to a lower equilibrium real interest rate and a higher equilibrium per capita capital stock. The equilibrium per capita capital stock is found, from (18), to be approximately 10.047595 compared with 2.213988 in the risk neutral model.

The implied risk premium that firms need to pay households is defined by (23). For current parameter values, the equilibrium risk premium works out to approximately 0.0000607. This is very small relative to the risk free rate. An endogenous rate of time preference appears incapable of producing a high risk premium. We noted above, however, that $\Gamma > \gamma$, so that households behave in a more risk averse manner when they have an endogenous rate of time preference. We also noted that while $B \cong 66.429631$, the expected discounted value of labor income, $Ey/r \cong 93.863282$. Therefore, households also appear risk averse in that they effectively discount uncertain labor income at a rate that exceeds the riskless rate of interest. Our risk premium therefore may be small because the risks we are pricing are unrealistically small compared with the risks associated with actual equity investments.[21]

5.2 Short-selling Constraints

Using the same parameter values, we imposed the constraints $1 \geq \rho \geq 0$ and $W_{ij} \geq 0$. The numerical algorithm we used was similar to the unconstrained model, although we needed to allow for corners when solving the first order conditions. Also, while the solution to the functional equation (34) for $V(W)$ is continuous with a continuous first derivative, $V''(W)$ can be discontinuous at wealth levels where the various constraints switch from being binding to becoming non-binding.[22] Specifically, we modified the algorithm as follows:

1. We approximated $\log(-V(W))$ in two regions of W space. The upper boundary of the first region was the maximum wealth level where a constraint was binding. In this first region, the *second* derivative of the spline approximation to $\log(-V(W))$ was allowed to be discontinuous at the constraint boundaries. In the second region, the spline approximation had a smooth second derivative, as in the unconstrained algorithm discussed above.[23]
2. We chose a finer grid of values for W close to zero. Even so, some discontinuity points in $V''(W)$ in

21. As we noted above, the results obtained by Aiyagari (1994) suggest that serial correlation in the idiosyncratic income shocks could also raise the risk premium.

22. The first derivative of $V(W)$, and the optimal solutions for $\rho(W)$ and $c_{ij}(W)$, can have "kinks" at the boundaries of the regions where the constraints are binding.

23. Continuity of $V(W)$ and $V'(W)$ at the interior boundary of the two regions was ensured by including a cubic "link" polynomial to evaluate $V(W)$ and $V'(W)$ for W values above the upper limit of the first region and the next highest W value, which was the lower limit of the second region.

the first region were close together, resulting in few points being available to determine some spline segments. Hence, the spline approximation in this region used polynomials of degree 4. The second region used polynomials of degree 6 as in the unconstrained model.
3. We also based the spline approximations in both regions on polynomials in log(30+W). In the unconstrained case, we used polynomials in log(B+W) and determined B iteratively.
4. The first order conditions (35)–(38) and (40) were first solved assuming (various subsets of) the constraints were binding. The values for the Lagrange multipliers were then calculated from (35)–(38) and (40). For wealth levels where the multipliers were positive we have a solution to the first order conditions. For the remaining wealth levels, the solution for the endogenous variables is interior and the corresponding multipliers are zero. The relevant first order conditions are then solved for maximizing values of those endogenous variables.

The final household value function

Figure 7 graphs the difference between the spline approximations to $V(W)$ in the unconstrained and constrained cases evaluated at the grid of W values used in the constrained case.

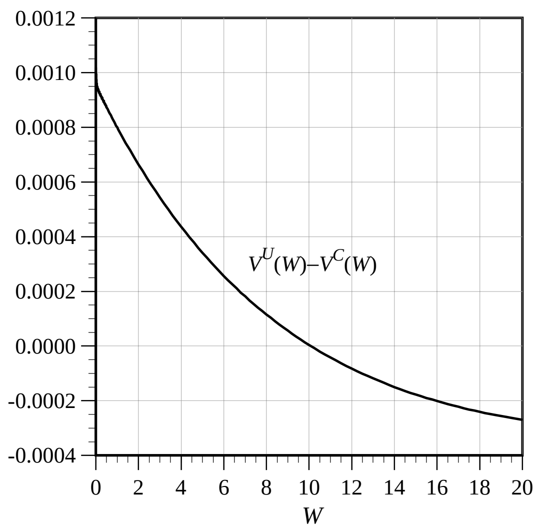

FIGURE 7. Difference between unconstrained and constrained utility

Figure 8 graphs $V(W)$ and the approximations to $V'(W)$, $V''(W)$ and $-V'(W)/V''(W)$ in the constrained case.[24] $V''(W)$ has been graphed only around its discontinuity points. If this were not done, the initial range of large negative values appears as a vertical line. We conclude:

Result 11. The short-selling constraints make households worse off at low wealth levels but high wealth

24.In the region where the constraints bind, the maximum absolute difference between the final value for $V(W)$ and the spline approximation is 9.62×10^{-8}, while the average absolute error over 262 values is 5.03×10^{-9}.

households are better off. The percentage differences are, however, very small.

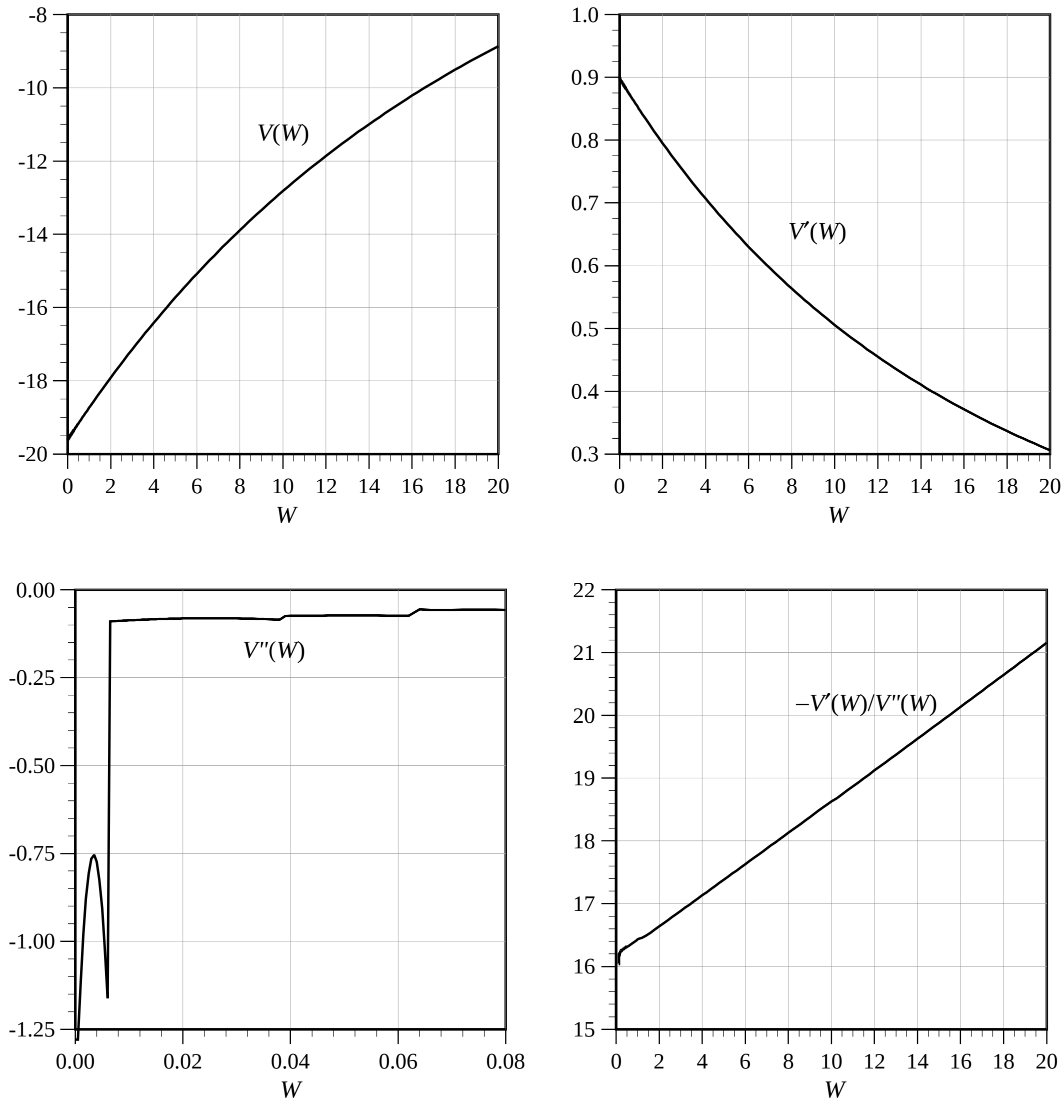

FIGURE 8. Value function in the constrained economy

Result 12. Short-selling constraints increase the degree of absolute risk aversion of the indirect utility of wealth function at low wealth levels where the constraints bind, but have very little effect on the concavity of $V(W)$ at higher wealth levels where they are non-binding.

Portfolio allocation

The first graph in Figure 9 plots the maximizing $\rho(W)$ in the economy with short-selling constraints. This function is similar to the truncated version of $\rho(W)$ in the unconstrained economy (graphed in the left panel of Figure 4). The third graph in Figure 9 plots the change in the value of the portfolio in states 0 (low aggregate income) and 1 (high aggregate income). From these two graphs we conclude:

Result 13. Short-selling constraints affect portfolio allocations only when the constraints bind. In particular, households hold less risky portfolios as W increases.

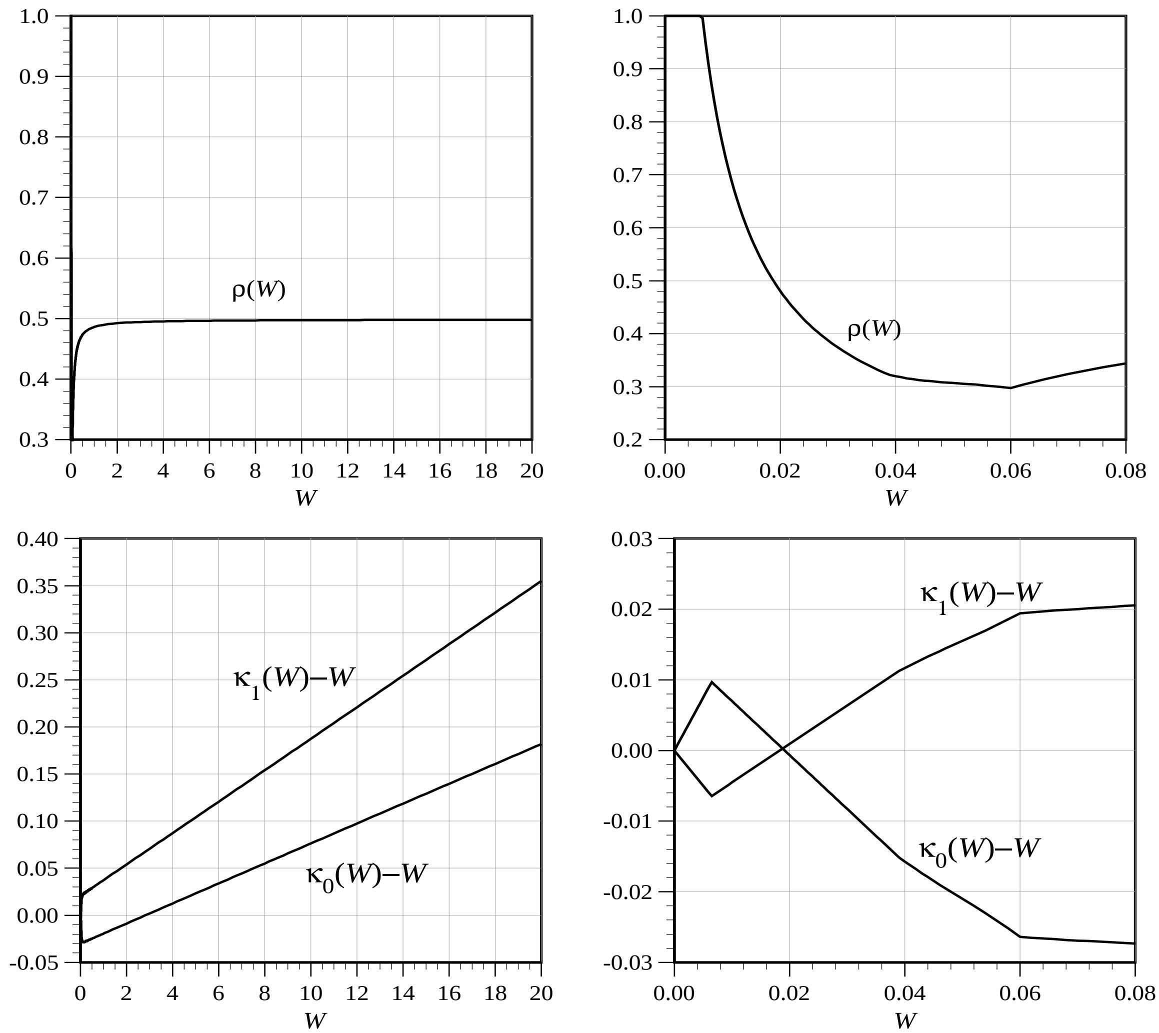

FIGURE 9. Portfolio allocation in the constrained economy

Finally, Figure 9 also illustrates optimal portfolio allocations at very low wealth levels.

Result 14. When W is very low, the short-selling constraints prevent households from distributing more income to the low aggregate labor income state 0. The constraint $\rho \le 1$ is binding for W very close to zero. Since at these very low wealth levels, all wealth is invested in claims that pay off only in state 0, the final graph shows that $W_{t+1} = \kappa_0(W_t) > W_t > \kappa_1(W_t) = 0$.

Result 15. At most wealth levels above those where the constraint $\rho \le 1$ *is binding, portfolios are biased toward assets that pay off in the high aggregate labor income state* 1. The higher return on the more risky assets that pay off only in state 1 makes them attractive to households once the constraints on consumption in the low labor income state are no longer binding.

Consumption

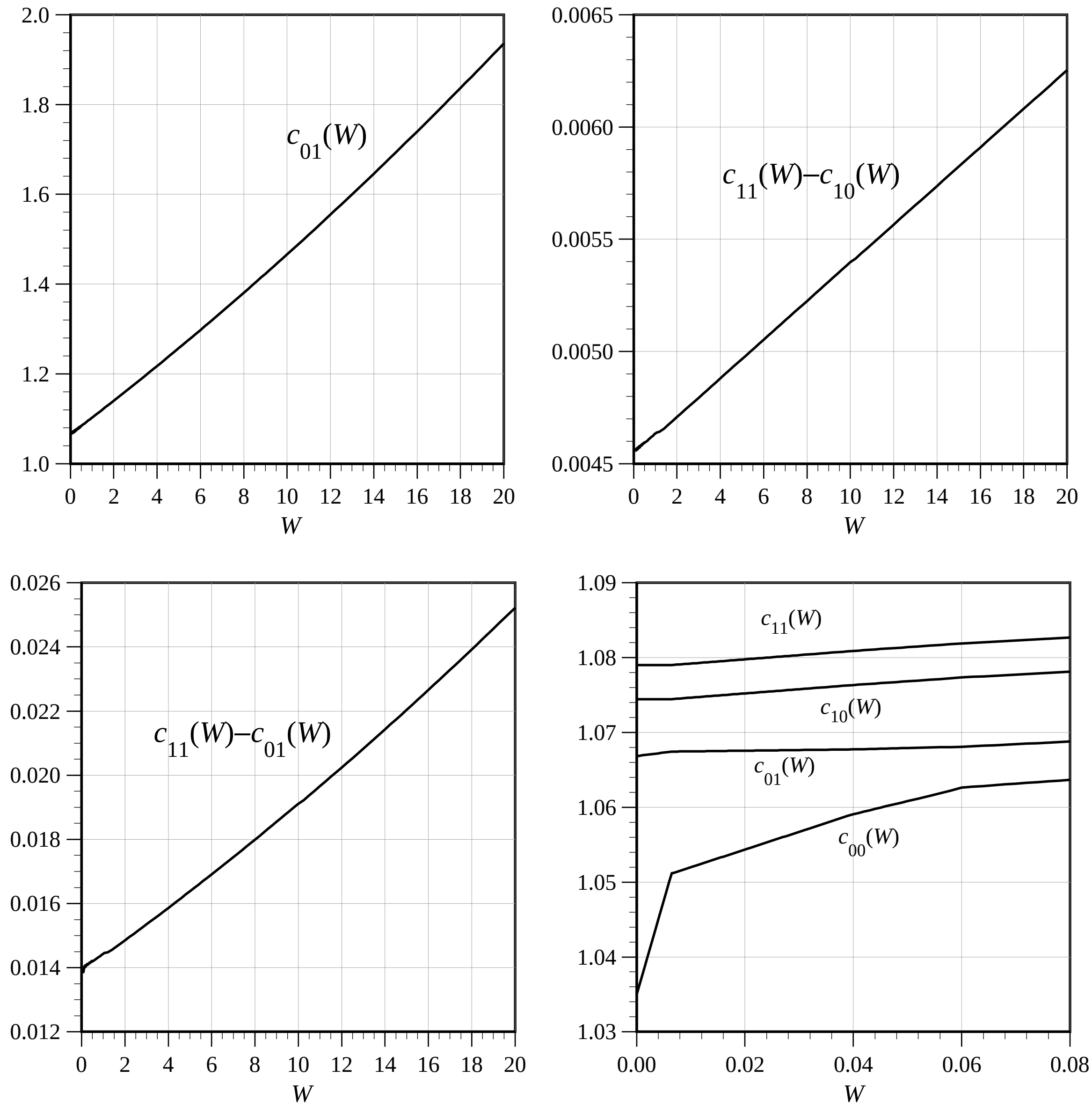

FIGURE 10. Consumption in the constrained economy

Figure 10 graphs maximizing consumption when households face short-selling constraints. We have plotted $c_{01}(W)$ rather than $c_{00}(W)$ in the first graph in Figure 10 since $c_{01}(W)$ is less affected by the constraints. The second graph in Figure 10 plots the disparity in consumption across *idiosyncratic* states as a function of household wealth. The third graph plots the disparity in consumption across *aggregate* states as a function of household wealth. The final graph in Figure 10 focuses on the maximizing consumption functions in the initial range of wealth levels where the constraints are binding.

Result 16. Short-selling constraints affect consumption only at wealth levels where the constraints bind. The first three graphs in Figure 10 resemble the graphs in Figure 5 for $W > 0$.

Result 17. The constraint $\rho \leq 1$ *ceases to bind at wealth levels considerably below those where the constraint* $W_{00} \geq 0$ *ceases to bind.* This is illustrated in the final graph in Figure 10. While the constraint $\rho \leq 1$ binds, consumptions in the high aggregate income state 1 equal current labor income and are independent of W. Consumption in the low aggregate, low idiosyncratic state (0,0) is most constrained and benefits most from increasing financial wealth. Even after $\rho(W)$ falls below 1, consumption in the (0,0) state remains constrained by $W_{00} \geq 0$. Asset as well as labor income is completely consumed in the (0,0) state while the constraint $W_{00} \geq 0$ is binding.

The wealth distribution

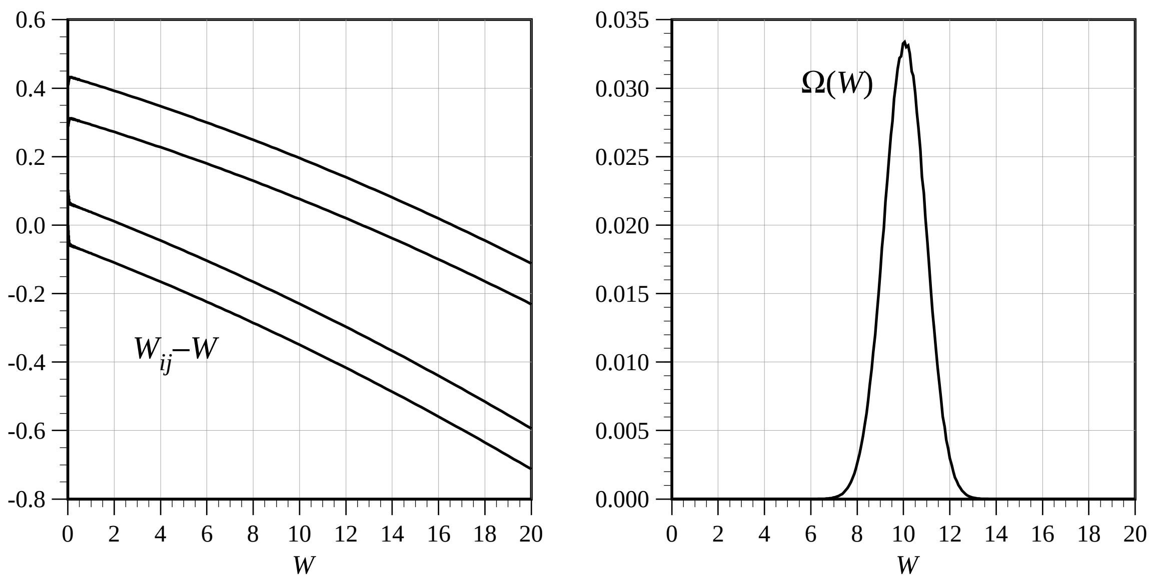

FIGURE 11. Stochastic difference equation for wealth in the constrained economy

Figure 11 shows that each of the mappings $W \to W_{ij}(W)$ has a single fixed point W_{ij}^*. At wealth levels where the constraints no longer bind, the mappings are very similar to the mappings in the unconstrained economy, and the fixed points are virtually identical. It is then not surprising that we have:[25]

Result 18. The cross sectional wealth distribution is very similar in the two economies.

Asset prices

Approximate solutions for the asset prices were $p_0 = 0.3997933$ and $p_1 = 0.586169$. By comparison,

25.Note that the number of bins, and the width of each bin, are different in Figure 11 and Figure 6.

the asset prices in the unconstrained case were $p_0 = 0.3997927$ and $p_1 = 0.586180$. Using equation (21), the implied risk-free interest rate $r \cong 0.0142375$ compared with 0.0142271 in the unconstrained case. The implied risk premium is approximately 0.00006075 compared with 0.00006068 in the unconstrained case. These asset price differences should be interpreted with caution, however, since they are near the margin of error inherent in our numerical approximation techniques.

6. Conclusion

We have examined the aggregate supply of assets by firms, the portfolios of heterogeneous households, and asset prices in an intertemporal production economy with aggregate and individual uncertainty. In order to obtain an equilibrium distribution of wealth across households, we made household rates of time preference a function of wealth. We also examined the effects of uninsured idiosyncratic risk and household borrowing constraints in the presence of production by firms and unrestricted trading by financial intermediaries. We showed that firm financial policies are irrelevant even when uninsurable income shocks and short-selling (or borrowing) constraints make capital markets incomplete for consumers. Modigliani-Miller leverage irrelevance holds because no single firm can alter the *aggregate* set of securities supplied to households in each risk class.

We found that households demand *less* risky asset portfolios as their financial wealth increases. This follows from the fact that indirect utility depends upon total wealth, which includes the capitalized value of expected labor income. As financial wealth increases, it becomes a larger proportion of total wealth, and thereby imparts more variability to household income. In order to achieve the desired variability of consumption, households reduce the riskiness of their portfolios at higher wealth levels.

Endogenous rates of time preference in the presence of household risk aversion, and uninsured idiosyncratic risk were shown to reduce the equilibrium real rate of interest. Households accumulate assets that have such low rates of return because they want to “self-insurance” against income fluctuations. We also showed that households are able to insure more effectively against idiosyncratic income shocks because such shocks are diversifiable while the aggregate income shocks are not.

While the riskless real rate of interest was low relative to household rates of time preference, the risk premium was also very low. It remains an open question whether the risk premium will remain low for other parameter values, including different specifications for the dependence of the time discount rate on past or future consumption or different distributions for the uninsured idiosyncratic risk.

We found that borrowing constraints affected the consumption and portfolio choices of low wealth households who would otherwise short assets. However, the constraints had little impact on most households, the distribution of wealth or equilibrium asset prices. We speculate that the flexibility provided by endogenous asset supplies greatly reduces the impact of borrowing constraints in our model.

Our model needs to be extended to incorporate rational expectations about future asset price variability in response to aggregate shocks. We conjecture that the *means* of the resulting stationary distributions for asset prices will not differ much from the constant values we have calculated in this paper. We suspect that extending the model to include liquid assets either in the form of inside or outside money might enable it to account for more of the evidence on household portfolio demands. Another important extension will be to introduce taxes. Multi-period models provide the appropriate setting in which to examine the intertemporal dimension of tax distortions including, for example, the effect of levying capital gains taxes on realized rather than accrued income.

7. Appendix

Following Epstein and Zin (1989) or Epstein (1992) we now assume household utility is recursive[26]

$$V_t = F[\mu(\tilde{c}_t), \mu(\tilde{V}_t)] \tag{76}$$

where $\mu(\tilde{x})$ is the "certainty equivalent" of the random variable x and F is called an "aggregator function," since it aggregates current consumption with an index of the future to determine current utility. We shall only consider the convenient functional forms

$$F(y, z) = \left[(1-\beta)y^{\frac{(\sigma-1)}{\sigma}} + \beta z^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{(\sigma-1)}} \tag{77}$$

where $0 < \beta < 1$ is the time discount factor, $\sigma > 0$ is the elasticity of substitution between current consumption and future utility, and

$$\mu(\tilde{x}_t) = [E_t \tilde{x}_t^{1-\gamma}]^{\frac{1}{(1-\gamma)}} \tag{78}$$

where $\gamma > 0$ equals the degree of relative risk aversion with respect to timeless gambles. Substituting (77) and (78) into (76), we can write current utility for our particular functional forms

$$V_t = \left[(1-\beta)[E_t \tilde{c}_t^{1-\gamma}]^{\frac{(\sigma-1)}{\sigma(1-\gamma)}} + \beta[E_t \tilde{V}_{t+1}^{1-\gamma}]^{\frac{(\sigma-1)}{\sigma(1-\gamma)}}\right]^{\frac{\sigma}{(\sigma-1)}} \tag{79}$$

Aversion toward consumption gambles increases as γ increases. In the special case where

$$1-\gamma = \frac{\sigma-1}{\sigma} \text{ that is } \gamma = \frac{1}{\sigma}$$

26.In our formulation, utility is defined *prior* to the resolution of uncertainty in period t. Epstein (1992) defines utility recursively when current consumption is known but next period's state is still uncertain.

household utility (76) will satisfy

$$V_t = [(1-\beta)E_t\tilde{c}_t^{1-\gamma} + \beta E_t\tilde{V}_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} = \left[(1-\beta)E_t\sum_{k=0}\beta^k\tilde{c}_{t+k}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}. \tag{80}$$

Risk neutrality with respect to consumption gambles within a period will apply as $\gamma \to 0$. If both $\gamma \to 0$ and $\sigma \to \infty$ consumer preferences will be both risk neutral and time additively separable. In the numerical analysis we only examined solutions for $\gamma > 1$ and $\sigma > 1$.

Definition: For recursive household preferences as specified in (79) the household *value function* $V(W)$ is the solution to the functional equation:

$$V(W) = \max_{\rho, c_{ij}, W_{ij}}\left[(1-\beta)[Ec_{ij}^{1-\gamma}]^{\frac{(\sigma-1)}{\sigma(1-\gamma)}} + \beta[EV(W_{ij})^{1-\gamma}]^{\frac{(\sigma-1)}{\sigma(1-\gamma)}}\right]^{\frac{\sigma}{(\sigma-1)}} \tag{81}$$

where

$$Ec_{ij}^{1-\gamma} = \pi\theta c_{00}^{1-\gamma} + \pi(1-\theta)c_{01}^{1-\gamma} + (1-\pi)\theta c_{10}^{1-\gamma} + (1-\pi)(1-\theta)c_{11}^{1-\gamma} \tag{82}$$

$$\begin{aligned} EV(W_{ij})^{1-\gamma} &= \pi\theta V(W_{00})^{1-\gamma} + \pi(1-\theta)V(W_{01})^{1-\gamma} \\ &+ (1-\pi)\theta V(W_{10})^{1-\gamma} + (1-\pi)(1-\theta)V(W_{11})^{1-\gamma} \end{aligned} \tag{83}$$

and the maximizations are carried out assuming p_0 and p_1 are constant. Given the numerical difficulties we encountered with this specification of utility, we decided to examine only the interior solution.

Theorem 7: For the maximization problem (81), consumptions and end of period assets in each state and the household portfolio allocation $\rho(W)$ satisfy the first order conditions:

$$(1-\beta)c_{00}^{-\gamma}(Ec_{ij}^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} = \beta V(W_{00})^{-\gamma}V'(W_{00})(EV(W_{ij})^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} \tag{84}$$

$$(1-\beta)c_{01}^{-\gamma}(Ec_{ij}^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} = \beta V(W_{01})^{-\gamma}V'(W_{01})(EV(W_{ij})^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} \tag{85}$$

$$(1-\beta)c_{10}^{-\gamma}(Ec_{ij}^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} = \beta V(W_{10})^{-\gamma}V'(W_{10})(EV(W_{ij})^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} \tag{86}$$

$$(1-\beta)c_{11}^{-\gamma}(Ec_{ij}^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} = \beta V(W_{11})^{-\gamma}V'(W_{11})(EV(W_{ij})^{1-\gamma})^{\frac{\sigma\gamma-1}{\sigma(1-\gamma)}} \tag{87}$$

$$\frac{\pi[\theta c_{00}^{-\gamma} + (1-\theta)c_{01}^{-\gamma}]}{p_0} = \frac{(1-\pi)[\theta c_{10}^{-\gamma} + (1-\theta)c_{11}^{-\gamma}]}{p_1} \tag{88}$$

and the budget constraints

$$W_{00} + c_{00} = y_0 + z_0 + \frac{\rho W}{\rho p_0 + (1-\rho) p_1} \tag{89}$$

$$W_{01} + c_{01} = y_0 + z_1 + \frac{\rho W}{\rho p_0 + (1-\rho) p_1} \tag{90}$$

$$W_{10} + c_{10} = y_1 + z_0 + \frac{(1-\rho) W}{\rho p_0 + (1-\rho) p_1} \tag{91}$$

$$W_{11} + c_{11} = y_1 + z_1 + \frac{(1-\rho) W}{\rho p_0 + (1-\rho) p_1} \tag{92}$$

Proof: With no constraints, the first order conditions are analogous to (35)–(39) with $\varphi_{ij} = \mu_i = 0$.

Now observe that for a value function V that is concave, V' is monotonic decreasing and the conditions (84)–(92) imply that c_{ij} and W_{ij} will be monotonic increasing in W. Also, if we define the initial asset holdings

$$A_0(W) = \frac{\rho W}{\rho p_0 + (1-\rho) p_1} \tag{93}$$

$$A_1(W) = \frac{(1-\rho) W}{\rho p_0 + (1-\rho) p_1} \tag{94}$$

for any given level of W, $c_{ij}(W) < c_{kl}(W)$ if and only if $y_i+z_j+A_i(W) < y_k+z_l+A_k(W)$. We examined the solution for parameter values which implied that $y_0+z_0 < y_1+z_0 < y_0+z_1 < y_1+z_1$.

As W declines, consumption levels will decline monotonically. From (88), the assumption that $\gamma > 0$ and the conclusion that $c_{10}(W) < c_{11}(W)$ for all W, observe that if $c_{00}(W) \to 0$, then $c_{10}(W) \to 0$. Further, if we are to ensure $c_{ij}(W) \geq 0$ for all values of W, and $c_{00}(W) \to 0$ as $W \to W_m$, we require $W_{ij}(W) \geq W_m$ for all $W \geq W_m$.

We examined the solution for the parameter values in Table 3. There are two major differences

TABLE 3. Parameter values for the recursive utility model

Parameter	β	γ	σ	π	θ	z_0	z_1	ε_0	ε_1	α	δ
Value	0.95	1.75	1.5	0.4	0.2	-0.75	0.1875	0.85	1.1	0.25	0.1

between these parameter values and the ones examined earlier. In initial experimentation with different parameter values, we discovered that *household behavior is extremely sensitive to the size of the idiosyncratic income shocks relative to the aggregate income shocks*.[27] In particular, the sensitivity of end-of-period wealth levels to increases in W increased as idiosyncratic shocks were made relatively more important. In turn, higher values of $W_{ij}'(W)$ aided the numerical analysis by decreasing the range of wealth levels we needed to consider. Thus, we increased the size of the idiosyncratic income shocks, and decreased the size of the aggregate income shocks by raising the capital depreciation rate δ.

For the parameter values in Table 3, and for asset all prices p_0 and p_1 that lead to approximate equality in per capita asset supplies and demands, we found that $W_{00}(W)-W$ was monotonically decreasing, and, for $i,j \neq 0$, $W_{ij}(W)-W_{00}(W)$ was monotonically increasing, in W. Further, for all W such that $c_{00}(W) > 0$, we found $W_{00}(W) < W_{ij}(W)$ for $i,j \neq 0$ and $W_{00}(W) < W$. We conclude that as $W \to W_m$ we must also have $W_{00}(W) \to W_m$. Also, from our observations above, we must have $c_{10}(W) \to 0$ and $W_{10}(W) \to W_m$ as $W \to W_m$. We conclude that W_m and $\rho(W_m)$ must satisfy the equations

$$0 = y_0 + z_0 + \frac{\rho(W_m)W_m}{\rho(W_m)p_0 + (1-\rho(W_m))p_1} \tag{95}$$

$$0 = y_1 + z_0 + \frac{(1-\rho(W_m))W_m}{\rho(W_m)p_0 + (1-\rho(W_m))p_1} \tag{96}$$

that is,

$$\rho(W_m) = \frac{(y_0-y_1)p_1 - (y_0+z_0)}{(y_0-y_1)(1+p_1-p_0) - 2(y_0+z_0)} \tag{97}$$

$$W_m = y_0 + z_0 + \frac{\rho(W_m)(y_0-y_1)}{1-2\rho(W_m)} \tag{98}$$

Finally, observe that since γ and $\sigma > 1$, (84) and (86) imply $V(W_{00}) \to 0$ and $V(W_{10}) \to 0$ as $W \to W_m$. For the parameter values in Table 3, and the range of asset prices we encountered, we found that W_m was approximately -2.5. We used sixth order spline functions in W, with derivatives up to the fourth constrained to be continuous across the break points, to approximate (W)) on a grid of W values from -2 to 250. The spline approximation was augmented by an initial quartic polynomial defined on $[W_m,-2]$ that is constrained to equal 0 at W_m and to equal the spline approximation, and have identical first, second and third derivatives to the spline approximation, at $W = -2$.

The numerical algorithm we used to solve the model in this case can be summarized as follows.

1. Given p_0 and p_1, solve for k from (18), y_0 and y_1 from (25) and W_m and $\rho(W_m)$ from (97) and (98).
2. For a spline approximation to $V_1(W)$ on [-2,250] solve for the quartic polynomial on $[W_m,-2]$.
3. Solve (84)–(88) for $c_{ij}(W)$ and $\rho(W)$ at the fixed grid of values for W (which differ from the breakpoints for the spline approximation). W_{ij} are eliminated from (84)–(88) using (89)–(92).

27. This has the very interesting implication that the values taken by *aggregate* variables in this model economy are *extremely sensitive* to the properties of the idiosyncratic income shocks.

4. Calculate $W_{ij}(W)$ from (89)–(92) and the maximizing $c_{ij}(W)$ and $\rho(W)$.
5. Calculate $V_2(W)$ by iterating (81) with fixed values of $c_{ij}(W)$, $W_{ij}(W)$ and $\rho(W)$.
6. If $\|V_2(W)-V_1(W)\| > 5\times10^{-9}$, obtain a new spline approximation to $V_2(W)$ and return to step 2.
7. Obtain the stationary wealth distribution for the 4 difference equations $W_{ij}(W)$ from step 4. Use this distribution and $\rho(W)$ from step 4 to calculate per capita demands for asset income, $\bar{\kappa}_0$ and $\bar{\kappa}_1$. Per capita supplies are given by (9) and (10). Adjust p_0 and p_1 in proportion to the excess demands

$$\Delta p_i = f(\bar{\kappa}_i - x_i) \tag{99}$$

and return to step 1, unless the adjustment in both prices is less than 10^{-7}.

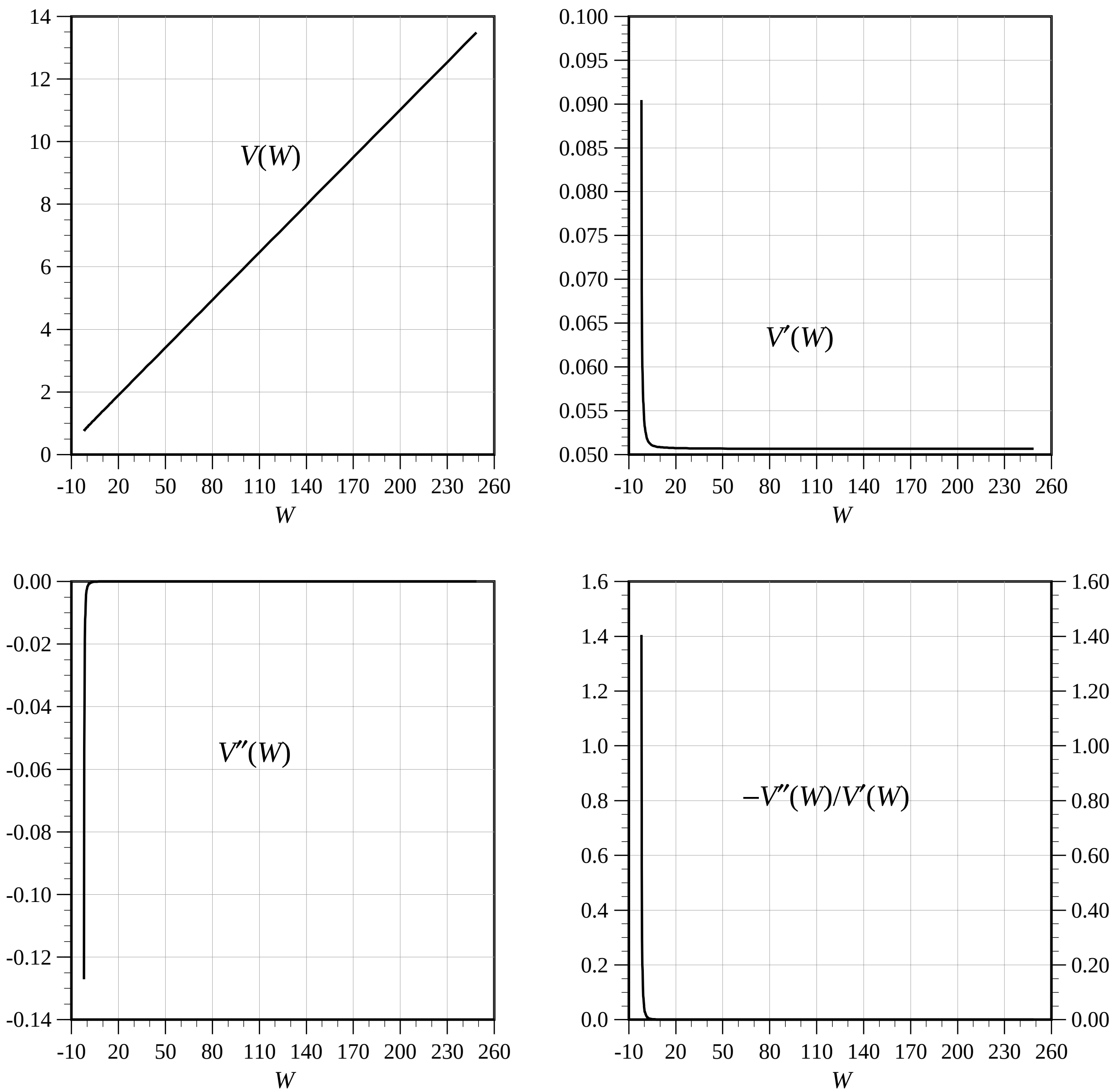

FIGURE 12. Value function with recursive utility and no short-selling constraints

For the parameter values in Table 3, the approximate equilibrium values of asset prices were $p_0 = 0.387702$ and $p_1 = 0.564192$ with W_m = -2.691275 and ρ_m = 0.480234. These asset prices yield a riskless real rate of interest of 0.050536 and a risk premium of 0.000275. The asset prices had a much larger effect on the maximizing $c_{ij}(W)$ and $\rho(W)$, and the approximation to $V(W)$, than was the case with the model discussed in the text. This made it difficult to find equilibrium asset prices and was the major reason we did not use this utility function for the main analysis in the paper.

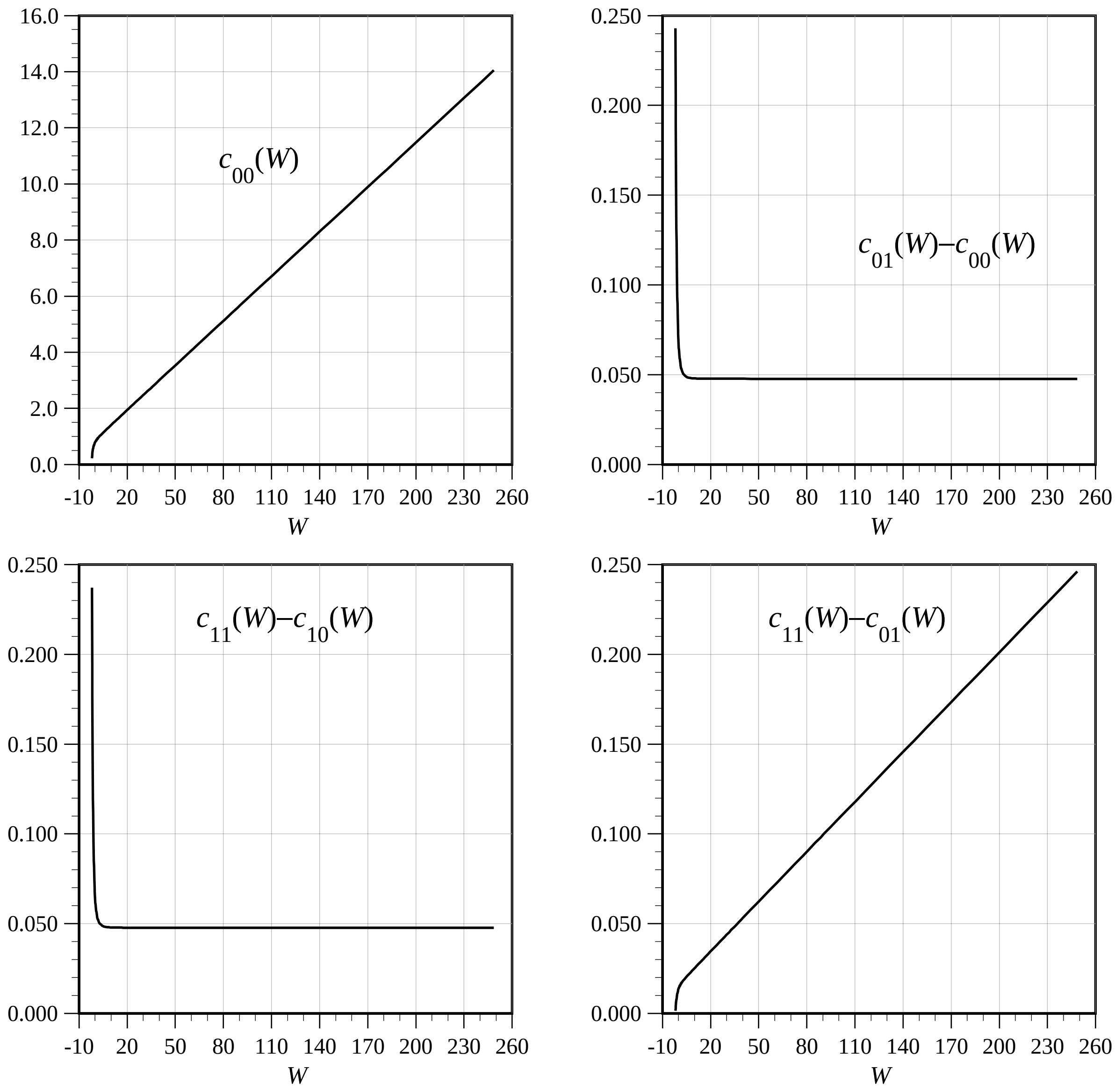

FIGURE 13. Consumption in the recursive utility model

The final approximation to the value function is graphed, along with its first and second derivatives and the coefficient of absolute risk aversion, in Figure 12. Except for W near W_m, V is close to linear. The non-linear segment occurs over a small range $W \in [W_m, -1]$ making it difficult to approximate V.

Consumption in the low income state, $c_{00}(W)$, is graphed in Figure 13 along with various differences in consumption across states. The idiosyncratic income shocks now have *less* of an effect on consumption as wealth increases. The income levels in the four states at the final asset prices were approximately y_0+z_0 = 0.004482, y_0+z_1 = 0.941982, y_1+z_0 = 0.226388 and y_1+z_1 = 1.163888. Thus, the difference in idiosyncratic incomes is 0.9375. Households evidently manage to obtain substantial self-insurance against these shocks, with the amount of insurance increasing with wealth.

Again, however, households are less effective at insuring against the aggregate income shocks. Since the consumption differences across aggregate states increases monotonically with wealth, the extent of self-insurance against these shocks declines as wealth increases. The graphs of consumption as a function of wealth also show how the idiosyncratic income shocks have a dramatic effect on household behavior at wealth levels close to the minimum level W_m.

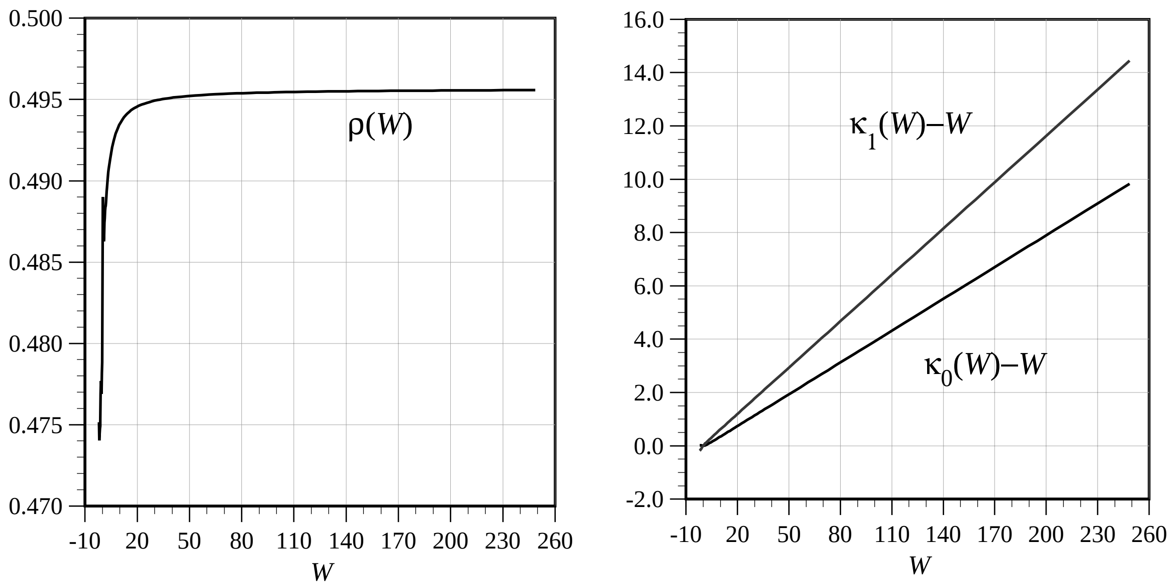

FIGURE 14. Portfolio allocation and asset income with recursive utility

Figure 14 shows that households again hold *less risky* portfolios as their wealth increases. The difference in asset income across the two aggregate income states nevertheless increases with wealth.

Finally, Figure 15 graphs the difference equations for the evolution of household wealth in the four states and the stationary cross-sectional wealth distribution implied by those difference equations. While the difference equations for the two idiosyncratic states behave very similarly at low wealth levels, the aggregate income shocks dominate behavior at higher wealth levels. Also, the two difference equations for the high aggregate income state do not appear to have upper fixed points. This would imply that there is no upper bound on household wealth. The tendency for wealth to decline in the low aggregate income state is so pronounced, however, that the stochastic difference equation nevertheless appears to produce a stationary cross-sectional wealth distribution. The second graph in

Figure 15 nevertheless shows that this distribution is skewed toward higher wealth levels.

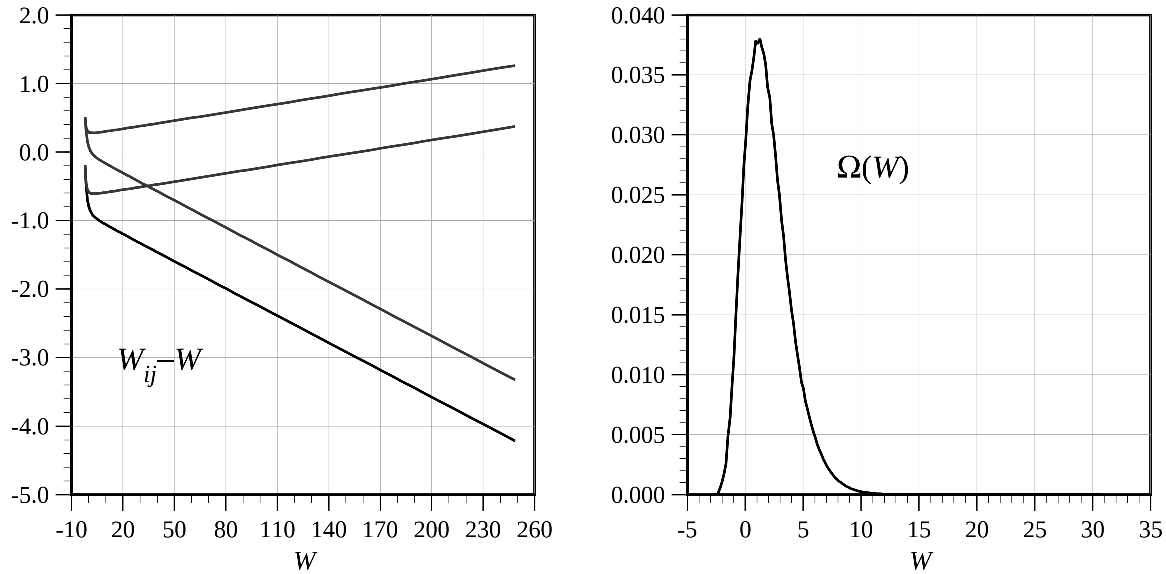

FIGURE 15. Stochastic difference equation with recursive utility

8. References

Aivazian, V.A. and Callen, J.L. (1987), "Miller's Irrelevance Mechanism: A Note," *Journal of Finance* 42 (March), pp. 169-180.

Aiyagari, S. Rao. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving." *Quarterly Journal of Economics* 109 (3), pp. 659-684.

Auerbach, A.J. and King, M.A. (1983), "Taxation, Portfolio Choice, and Debt-Equity Ratios: A General Equilibrium Model," *Quarterly Journal of Economics* 48 (November), pp. 587-609.

Becker, R.A., (1980), "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics*, Vol. 95, pp. 375-382.

Benninga, S. and Protopapadakis, A. (1990), "Leverage, Time Preference and the 'Equity Premium Puzzle'," *Journal of Monetary Economics*, Vol 25, No. 1, pp. 49-58.

Cochrane, J. and Hansen, L. (1992), "Asset Pricing Explorations for Macroeconomics," in *NBER Macroeconomics Annual 1992*, edited by O.J. Blanchard and S. Fischer, Cambridge, MA: MIT Press.

Constantinides G.M., (1990), "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, Vol. 98, No. 3, pp. 519-543.

Deaton, Angus (1991), "Saving and Liquidity Constraints," *Econometrica*, 59, pp. 1221-1248.

Dixit, A.K., (1987), "Trade and Insurance with Moral Hazard," *Journal of International Economics* 23, pp. 201-220.

Dixit, A.K., (1989), "Trade and Insurance with Adverse Selection," *Review of Economic Studies*, 56, pp. 235-247.

Dammon, R.M. (1988), "A Security Market and Capital Structure Equilibrium Under Uncertainty with Progressive Personal Taxes," *Research in Finance*, 7, pp 53-74.

Epstein L.G., (1987), "A Simple Dynamic General Equilibrium Model," *Journal of Economic Theory*, Vol. 41, pp. 68-75.

Epstein L.G., (1992), "Behavior under risk: recent developments in theory and applications," Chapter 1, *Advances in economic theory: Sixth World Congress, volume II*, edited by Jean-Jacques Laffont, Cambridge: Cambridge University Press.

Epstein L.G. and Hynes J.A., (1983), "The Rate of Time Preference and Dynamic Economic Analysis," *Journal of Political Economy*, Vol. 91, No. 4, pp. 611-635.

Epstein L.G. and Shi S., (1993), "Habits and Time Preference," *International Economic Review*, Vol. 34, No. 1, pp. 61-84.

Epstein L.G. and S.E. Zin, (1989), "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, Vol. 57, No. 4, pp.937-969.

Ferson W.E. and Constantinides G.M., (1991), "Habit Persistence and Durability in Aggregate Consumption," *Journal of Financial Economics*, Vol. 29, pp. 199-240.

Hartley P.R. (1994), "Interest Rates in a Credit Constrained Economy," *International Economic Review*, Vol. 35, No. 1 (February), pp. 23-60.

Hartley P.R. (1995), "Value Function Approximation in the Presence of Uncertainty and Inequality Constraints: An Application to the Demand for Credit Cards," forthcoming, *Journal of Economic Dynamics and Control.*

Heaton J. and Lucas D., (1992), "The Effects of Incomplete Insurance Markets and Trading Costs in a Consumption-Based Asset Pricing Model," *Journal of Economic Dynamics and Control*, Vol. 16, pp. 601-620.

Judd, Kenneth L., (1992), "Projection methods for solving aggregate growth models," *Journal of Economic Theory* 58, 410–452.

Kocherlakota N.R. (1990), "On the 'discount' factor in growth economies," *Journal of Monetary Economics*, Vol. 25, No. 1, pp. 43-47.

Lucas Robert E., Jr. and Stokey N.L., (1982), "Optimal Growth with Many Consumers," *Journal of Economic Theory*, Vol. 32, pp. 139-171.

Mehra R. and Prescott E.C., (1985), "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, Vol. 15, pp. 145-161.

Miller M.H. (1987), "Debt and Taxes," *Journal of Finance*, Vol. 32, pp. 261-275.

Obstfeld, M., (1990), "Intertemporal Dependence, Impatience, and Dynamics," *Journal of Monetary Economics*, Vol. 26, pp. 45-75.

Ryder H.E. and Heal G.M., (1973), "Optimum Growth and Intertemporally Dependent Preferences," *Review of Economic Studies*, Vol. 40, pp. 1-33.

Uzawa H., (1968), "Time Preference, the Consumption Function, and Optimum Asset Holdings," in J.N. Wolde, ed., *Capital and Growth: Papers in Honour of Sir John Hicks* (Chicago: Aldine)