Econ 400

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EXERCISE 2: Answers

1. Determine the order of magnitude of each of the following sequences of real numbers, as $t \to \infty$:
   a. $y_t = t$
      The order of a sequence of reals is given by the power of t *that yields a limit when divided into the sequence. For this sequence* $y_t/t = 1$ *has a limit of 1 so* $y_t = O(t)$.
   b. $y_t = 3 + t^2$. *Here,* $\lim\limits_{n\to\infty} y_t/t^2 = 1$, *so* $y_t = O(t^2)$.
   c. $y_t = 3 + 1/t^2$. *Here,* $\lim\limits_{n\to\infty} y_t = 3$, *so* $y_t = O(1)$.
   d. $y_t = t + t^{1/2}$. *Here,* $\lim\limits_{n\to\infty} y_t/t = 1$, *so* $y_t = O(t)$.
   e. $y_t = (1 + 2t + t^2)/(1 + 2t^2)$. *Since the* t^2 *terms dominate in the numerator and denominator, then* $\lim\limits_{n\to\infty} y_t = 1/2$, $y_t = O(1)$.

2. Suppose $x_t \sim i.i.d$, $E[x_t] = \mu$, and $V[x_t] = \sigma^2$ then determine the order in probability of the following sequences of random variables:
   a. $x_1 - \mu$
      The order in probability of a sequence of random variables is the power of n *that yields a sequence with bounded or stable limiting distribution. Here, since* x_1 *and hence* $x_1 - \mu$ *have a stable distribution already* $x_1 - \mu = O_p(1)$.
   b. $\bar{X}_n = \sum_{t=1}^n x_t/n$. *Rewriting* $\bar{X}_n = \mu + (\bar{X}_n - \mu) = O(1) + O_p(1/\sqrt{n}) = O_p(1)$, *where we use the result from (c) below.*
   c. $\bar{X}_n - \mu$. *Here since* x_t *is* $i.i.d.$, *and has first two moments then* $(\bar{X}_n - \mu)/(1/\sqrt{n}) = n^{1/2}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ *then* $\bar{X}_n - \mu = O_p(1/\sqrt{n})$.
   d. $(\bar{X}_n)^2 - \mu^2$. *Adding and subtracting we get* $((\bar{X}_n - \mu) + \mu)^2 - \mu_t^2 = (\bar{X}_n - \mu)^2 + 2\mu(\bar{X}_n - \mu) + \mu^2 - \mu^2 = (\bar{X}_n - \mu)^2 + 2\mu(\bar{X}_n - \mu)$. *Now the second term is* $O_p(1/\sqrt{n})$ *from (c) and the first must therefore be* $O_p(1/n)$, *so the second dominates and* $(\bar{X}_n)^2 - \mu^2 = O_p(1/\sqrt{n})$.
   e. $\sum_{t=1}^n (x_t - \mu)^2/n - \sigma^2$. *Define* $w_t = (x_t - \mu)^2$ *as an* $i.i.d.$ *random variable with expectation* σ^2 *and apply Khintchine's theorem to find* $\operatorname{plim}_{n\to\infty}\overline{w}_n = \sigma^2 \text{ which means } \overline{w}_n - \sigma^2 = o_p(1)$. *If we also assume a variance for* w_t *(fourth moment of* x_t*) and apply the results from (c) to this new variable then* $\sum_{t=1}^n (x_t - \mu)^2/n - \sigma^2 = \overline{w}_n - \sigma^2 = O_p(1/\sqrt{n})$.

3. Suppose $x_t \sim N(\mu, \sigma^2) \sim i.i.d.$
   a. Show that $\widetilde{X} = x_1 + 1/n$ is an asymptotically unbiased estimator of μ but is not consistent. *Now* $E[x_1] = \mu$ *so* $E[\widetilde{X}] = \mu + 1/n$ *so* $\lim\limits_{n\to\infty} E[\widetilde{X}] = \mu$ *and the estimator is asymptotically unbiased. But* $\widetilde{X} \backsim N(\mu + 1/n, \sigma^2) \xrightarrow{d} N(\mu, \sigma^2)$ *so the estimator's distribution is not collapsing around the target and is therefore not consistent.*
   b. Determine the limiting distribution of $[n^{1/2}(\bar{X}_n - \mu)]$. *Since the* x_t *are jointly normal any linear combination of them, including the average will also be normal also. Thus* $\bar{X}_n \backsim N(\mu, \sigma^2/n)$ *and* $n^{1/2}(\bar{X}_n - \mu) \backsim N(0, \sigma^2) \xrightarrow{d} N(0, \sigma^2)$.
   c. Show that $\bar{X}_n$ is asymptotically efficient relative to $\widetilde{X}$. *Both* $\widetilde{X}$ *and* $\bar{X}_n$ *are asymptotically unbiased and normal thus we choose the one with the smaller variance* $\bar{X}_n$.
   d. Determine the limiting distribution of $[n^{1/2}(\bar{X}_n + \widetilde{X}/n - \mu)]$. *Rewrite* $n^{1/2}(\bar{X}_n + \widetilde{X}/n -$

$\mu) = n^{1/2}(\bar{X}_n - \mu) + \widetilde{X}/\sqrt{n}$ *and note that the first is* $O_p(1)$ *while the second in* $O_p(1/\sqrt{n})$*. Thus* $[n^{1/2}(\bar{X}_n + \widetilde{X}/n - \mu)] \xrightarrow{d} N(0, \sigma^2)$*, which is the same as the limiting distribution of the first term.*

e. Determine the limiting distribution of $[(\bar{X}_n - \mu)/(s^2/n)^{1/2}]$ if $\underset{n\to\infty}{plim} s^2 = \sigma^2$. *Rewrite* $(\bar{X}_n - \mu)/(s^2/n)^{1/2} = (\bar{X}_n - \mu)/(\sigma^2/n)^{1/2} \cdot (\sigma^2/n)^{1/2}/(s^2/n)^{1/2}$*. The first component* $(\bar{X}_n - \mu)/(\sigma^2/n)^{1/2} \xrightarrow{d} N(0,1)$ *while the second* $(\sigma^2/n)^{1/2}/(s^2/n)^{1/2} \xrightarrow{p} 1$*, so* $(\bar{X}_n - \mu)/(s^2/n)^{1/2} \xrightarrow{d} N(0,1)$.

4. Suppose $x_t \sim i.i.d$ with $E[x_t] = \mu$ and $V[x_t] = \sigma^2$ but not necessarily normal

   a. Show that $\underset{n\to\infty}{plim} \bar{X}_n = \mu$. *Since* $x_t \sim i.i.d$ *with* $E[x_t] = \mu$ *then the sample mean must be consistent for the population mean by Khintchine's theorem. We can also use convervence in quadratic mean since a variance exists.*

   b. Prove that $s^2 = \sum_{t=1}^{n}(x_t - \bar{X}_n)^2/(n-1)$ is consistent for σ^2. *Rewrite* $s^2 = \sum_{t=1}^{n}(x_t - \bar{X}_n)^2/(n-1) = \frac{n}{n-1}[\frac{1}{n}\sum_{t=1}^{n}((x_t - \mu) + (\mu - \bar{X}_n))^2]$
   $= \frac{n}{n-1}[\frac{1}{n}\sum_{t=1}^{n}(x_t - \mu)^2 + \frac{2}{n}\sum_{t=1}^{n}(x_t - \mu)(\mu - \bar{X}_n) + (\mu - \bar{X}_n)^2]$. *The ratio* $\frac{n}{n-1} \to 1$*, the first term in brackets* $\frac{1}{n}\sum_{t=1}^{n}(x_t - \mu)^2 \xrightarrow{p} \sigma^2$ *by LLN and the last is* $O_p(1/n)$ *as is the second. Thus* $s^2 = \sum_{t=1}^{n}(x_t - \bar{X}_n)^2/(n-1) \xrightarrow{p} \sigma^2$.

   c. Determine the limiting distribution of $[n^{1/2}(\bar{X}_n - \mu_0)]$ under both $H_0 : \mu = \mu_0$ and $H_1 : \mu = \mu_1 = \mu_0 + \gamma/\sqrt{n}$. *Rewrite* $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) + n^{1/2}(\mu - \mu_0)$*. Under* H_0 *the second term is zero and* $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$*. Under* H_1 *the second term becomes* $n^{1/2}(\mu - \mu_0) = n^{1/2}(\gamma/\sqrt{n}) = \gamma$ *so* $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) + \gamma \xrightarrow{d} N(\gamma, \sigma^2)$*. We have a shifted normal asymptotically under the alternative.*

   d. Determine the limiting distribution of $[(\bar{X}_n - \mu)/(s^2/n)^{1/2}]$. *This the same as (3.e).*